CSCI 6214— Randomized Algorithms— FALL 2021
Problem Set #3

Problem 1:

Suppose that we have n jobs to distribute among m processors. For simplicity, we assume that m divides n. A job takes 1 step with probability p and $k > 1$ steps with probability $1-p$. Use Chernoff bounds to determine upper and lower bounds (that hold with high probability) on when all jobs will be completed if we randomly assign exactly n/m jobs to each processor.

Problem 2:

- Let $X_1, \cdots, X_n$ be independent Poisson trials such that $Pr(X_i) = p_i$ and let $a_1, \cdots, a_n$ be real numbers in $[0, 1]$. Let $X = \sum_{i=1}^{n} a_i X_i$ and $\mu = E[X]$. Then the following Chernoff bound holds: for any $\delta > 0$,

$$Pr(X \geq (1+\delta)\mu) \leq (\frac{e^{\delta}}{(1+\delta)^{1+\delta}})^{\mu}$$

  . Prove a similar bound for the probability that $X \leq (1-\delta)\mu$ for any $0 < \delta < 1$.

- Let $X_1, \cdots, X_n$ be independent random variables such that $Pr(X_i = 1 - p_i) = p_i$ and $Pr(X_i = -p_i) = 1 - p_i$. Let $X = \sum_{i=1}^{n} X_i$. Prove that

$$Pr(|X| \geq a) 2e^{-2a^2/n}.$$

  Hint: You may need to assume the inequality $p_i e^{\lambda(1-p_i)} + (1-p_i)e^{-\lambda p_i} \leq e^{\lambda^2/8}$. This inequality is difficult to prove directly.

Problem 3:

Recall that a function f is said to be convex if, for any x_1, x_2 and for $0 \leq \lambda \leq 1$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

- Let Z be a random variable that takes on a (finite) set of values in the interval $[0, 1]$ and let $p = E[Z]$. Define the Bernoulli random variable X by $Pr(X = 1) = p$ and $Pr(X = 0) = 1 - p$. Show that $E[f(Z)] \leq E[f(X)]$ for any convex function f.

- Use the fact that $f(x) = e^{tx}$ is convex for any $t \geq 0$ to obtain a Chernoff bound for the sum of n independent random variables with distribution Z as in part (a), based on a Chernoff bound for independent Poisson trials.