CSCI 6214— Randomized Algorithms— FALL 2021
Problem Set #4

Problem 1:

Consider throwing m balls into n bins, and for convenience let the bins be numbered from 0 to $n-1$. We say there is a $k-$ gap starting at bin i if bins $i, i+1, \cdots, i+k-1$ are all empty.

- Determine the expected number of k-gaps.
- Prove a Chernoff-like bound for the number of k-gaps. (Hint: If you let $X_i = 1$ when there is a k- gap starting at bin i, then there are dependencies between X_i and X_{i+1}; to avoid these dependencies, you might consider X_i and X_{i+k}.)

Problem 2:

Suppose that we vary the balls-and-bins process as follows. For convenience let the bins be numbered from 0 to $n-1$. There are $\log_2 n$ players. Each player randomly chooses a starting location l uniformly from $[0, n-1]$ and then places one ball in each of the bins numbered $l \mod n, l+1 \mod n, \cdots, l+n/\log_2 n - 1 \mod n$. Argue that the maximum load in this case is only $O(\log\log n/\log\log\log n)$ with probability that approaches 1 as $n \to \infty$.

Problem 3:

We consider another way to obtain Chernoff-like bounds in the setting of balls and bins without using the relationship between the real distribution and the Poisson distribution we saw in class. Consider n balls thrown randomly into n bins. Let $X_i = 1$ if the i-th bin is empty and 0 otherwise. Let $X = \sum_{i=1}^{n} X_i$. Let Y_i, $i = 1, \cdots, n$, be independent Bernoulli random variables that are 1 with probability $p = (1-1/n)^n$. Let $Y = \sum_{i=1}^{n} Y_i$.

- Show that $E[X_1 X_2 \cdots X_k] \le E[Y_1 Y_2 \cdots Y_k]$ for any $k \ge 1$.
- Show that $E[e^{tX}] \le E[e^{tY}]$ for all $t \ge 0$. (Hint: Use the expansion for e^x and compare $E[X^k]$ to $E[Y^k]$.)
- Derive a Chernoff bound for $Pr(X \ge (1+\delta)E[X])$.