CSCI 6214— Randomized Algorithms— FALL 2021
Problem Set #7

Problem 1:

Consider the MaxCut problem where each node has at most neighbors. Give the best poly-time probabilistic approximation algorithm you can. De-randomize that algorithm if possible. (hint:one approach is to consider a maximal matching in a graph, your ratio should be at least 5/9)

Problem 2:

A random variable $X \in \{0,1\}^n$ is called pairwise-independent if for all $1 \leq i < j \leq n$ and all $a, b \in \{0,1\}$, $Pr[X_i = a \quad \text{and} \quad X_j = b] = 1/4$. That is,the restriction of X to any 2 coordinates is uniformly distributed in $\{0,1\}^2$. A function $H : \{0,1\}^r \rightarrow \{0,1\}^n$ is pairwise independent if the random variable $X = H(U)$ is pairwise independent, where $U \in \{0,1\}^r$ is uniformly chosen. In this case, we say that X has seed length r. We will show a construction of such a H with seed length $r = \log n + O(1)$. H is called a "pseudorandom generator". Assume that $n = 2^k$. We will define a function $H : \{0,1\}^r \rightarrow \{0,1\}^n$, where we identify the coordinates of the output of H with $\{0,1\}^k$, which is the binary expansion of the coordinate. Let $U \in \{0,1\}^{k+1}$ be uniformly chosen. Write $u = u_1, \cdots, u_{k+1}$ where $u_i \in \{0,1\}$. Define $H(U) \in \{0,1\}^n$ as follows. For every $x \in \{0,1\}^k$, the x-coordinate of $H(U)$ is defined to be

$$H(U)_x = (\sum_{i=1}^{k} u_i x_i) + u_{k+1} \mod 2$$

- Prove that $H(U)$ is pairwise independent.
- Prove that for general n (not necessarily a power of 2) this can be used to give a pairwise independent random variable $X \in \{0,1\}^n$ with seed length $r = \log n + O(1)$.
- Prove that the construction is optimal: for any $H : \{0,1\}^r \rightarrow \{0,1\}^n$ which is pairwise independent, it must hold that $r \geq \log n$.

  Hint: consider the $2 \times n$ matrix $M_{u,i} = (-1)^{H(u)_i}$. Prove that its columns are pairwise orthogonal. Conclude that the columns must be linearly independent, and hence $2^r \geq n$.

Problem 3:

Recall the factor-2 randomized algorithm for MAXCUT in the book. For $x \in \{0,1\}^n$ define its associated set $S(x) = \{v_i : x_i = 1\}$.

- Prove that if $X \in \{0,1\}^n$ is chosen from a pairwise independent distribution then also $E_X[e(S(X))] = m/2$, where $e(S(X))$ are the cut edges.
- Combine this with the construction from problem 2, to give an alternative deterministic algorithm which finds a factor-2 approximation of the MAXCUT value.