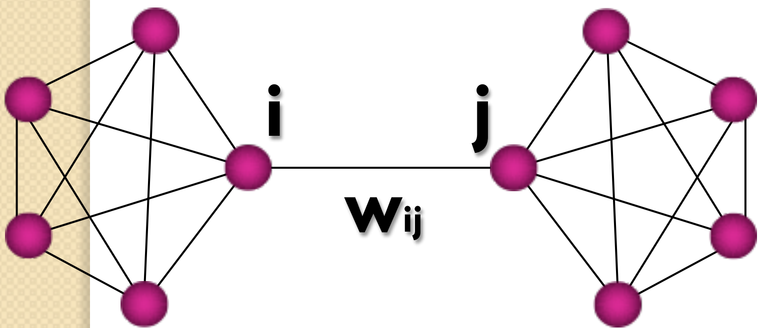
Computational Complexity. Lecture 11

Expansion and Eigenvalues

Alexandra Kolla

Representing Graphs

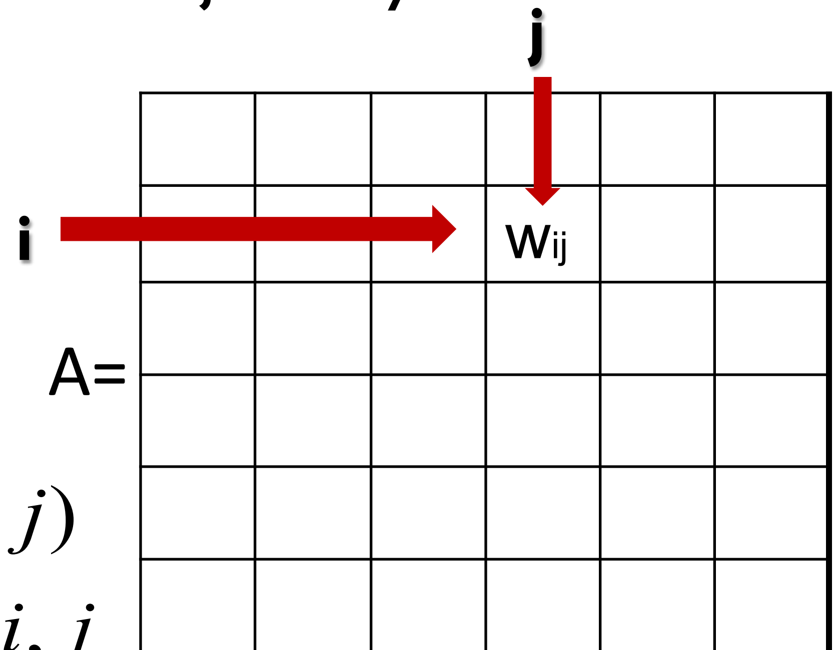


V: n nodes
E: m edges

G = {V,E}

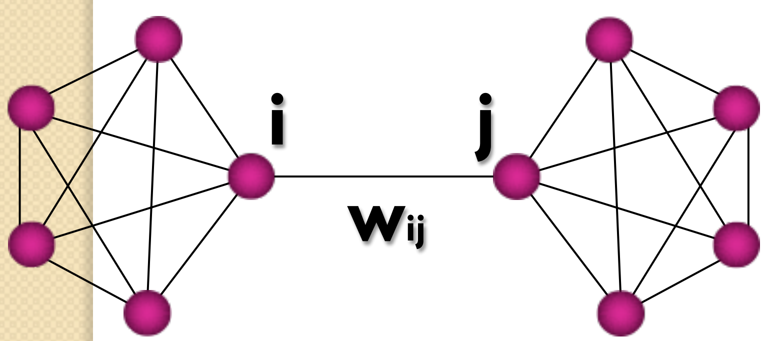
Obviously, we can represent a graph with an nxn matrix

Adjacency matrix

A=


$$A_{ij} = \begin{cases} w_{ij} & weight\ of\ edge\ (i, j) \\ 0 & if\ no\ edge\ between\ i, j \end{cases}$$

Representing Graphs



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V: n nodes
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What is not so obvious, is that once we have matrix representation view graph as **linear operator**

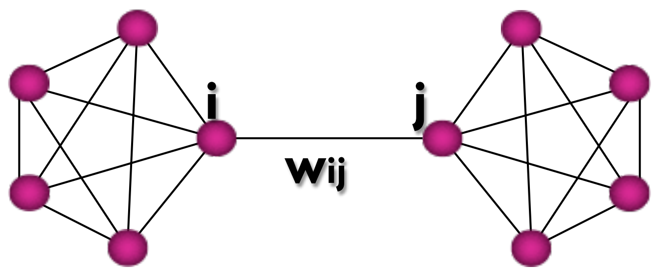
- Can be used to multiply vectors.
- Vectors that don't rotate but just scale = eigenvectors.
- Scaling factor= eigenvalue

$$A : \Re^n \rightarrow \Re^n$$

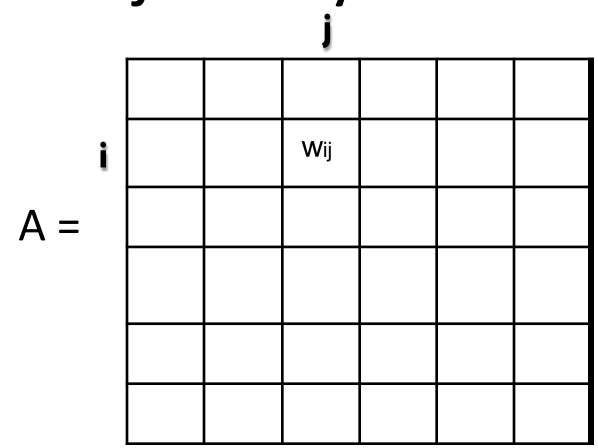
$$Ax = \mu x$$

Amazing how this point of view gives information about graph

"Listen" to the Graph



Adjacency matrix



List of eigenvalues

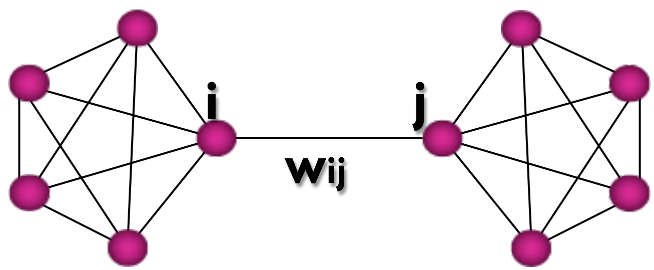
$\{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n\}$:graph SPECTRUM

Eigenvalues reveal global graph properties not apparent from edge structure

A drum:

Hear shape of the drum

“Listen” to the Graph



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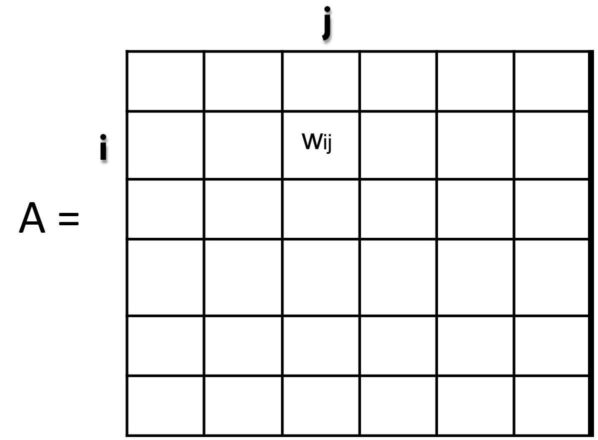

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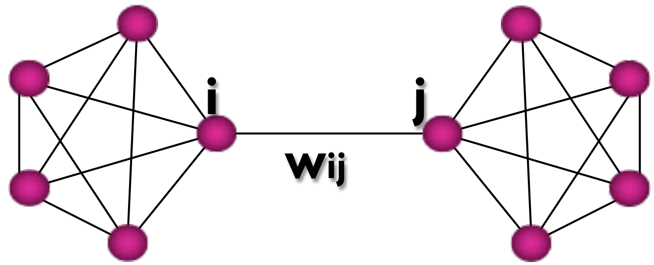
Hear shape of the drum

Its sound:

"Listen" to the Graph



Adjacency matrix

A =

		Wij			

List of eigenvalues

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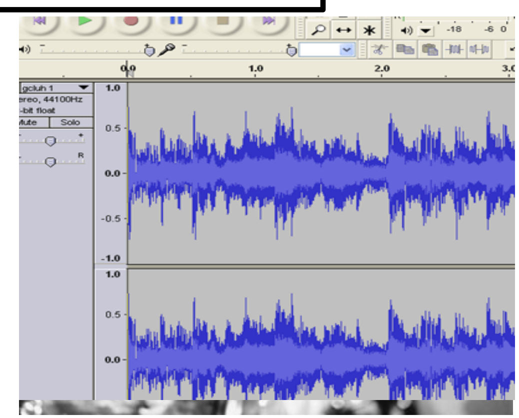
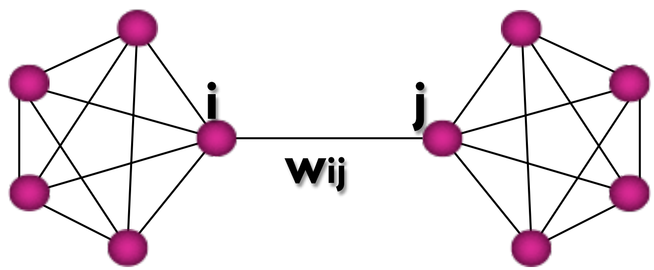
Eigenvalues reveal global graph properties not apparent from edge structure

Hear shape of the drum

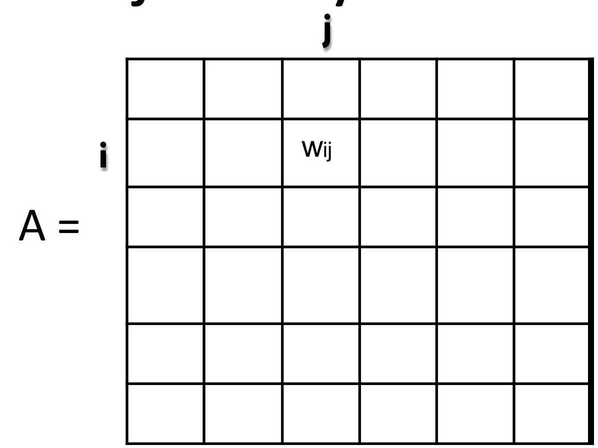
Its sound (eigenfrequenies):



"Listen" to the Graph



Adjacency matrix

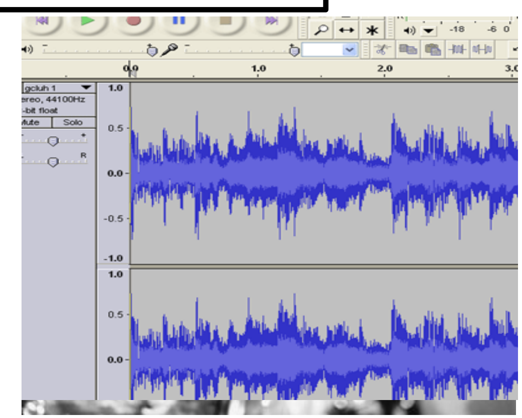


List of eigenvalues

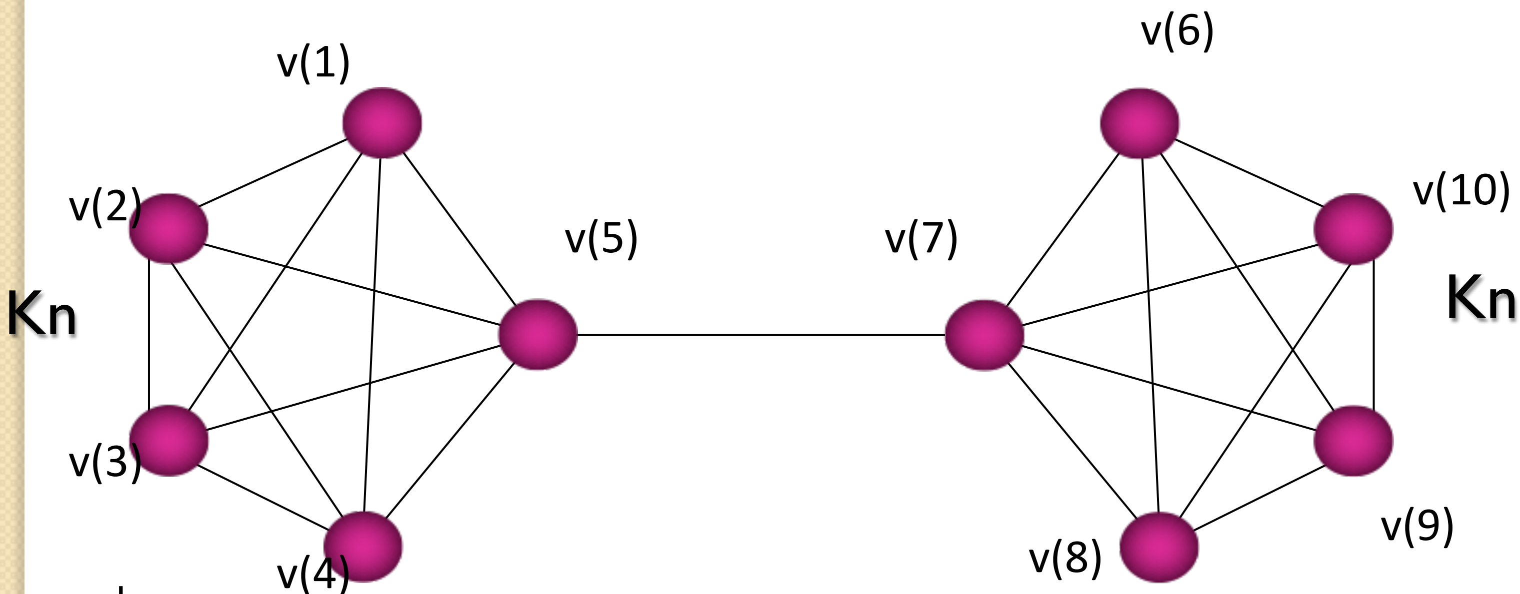
$\{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n\}$:graph SPECTRUM

Eigenvalues reveal global graph properties not apparent from edge structure

If graph was a drum, spectrum would be its sound



Eigenvectors are Functions on Graph



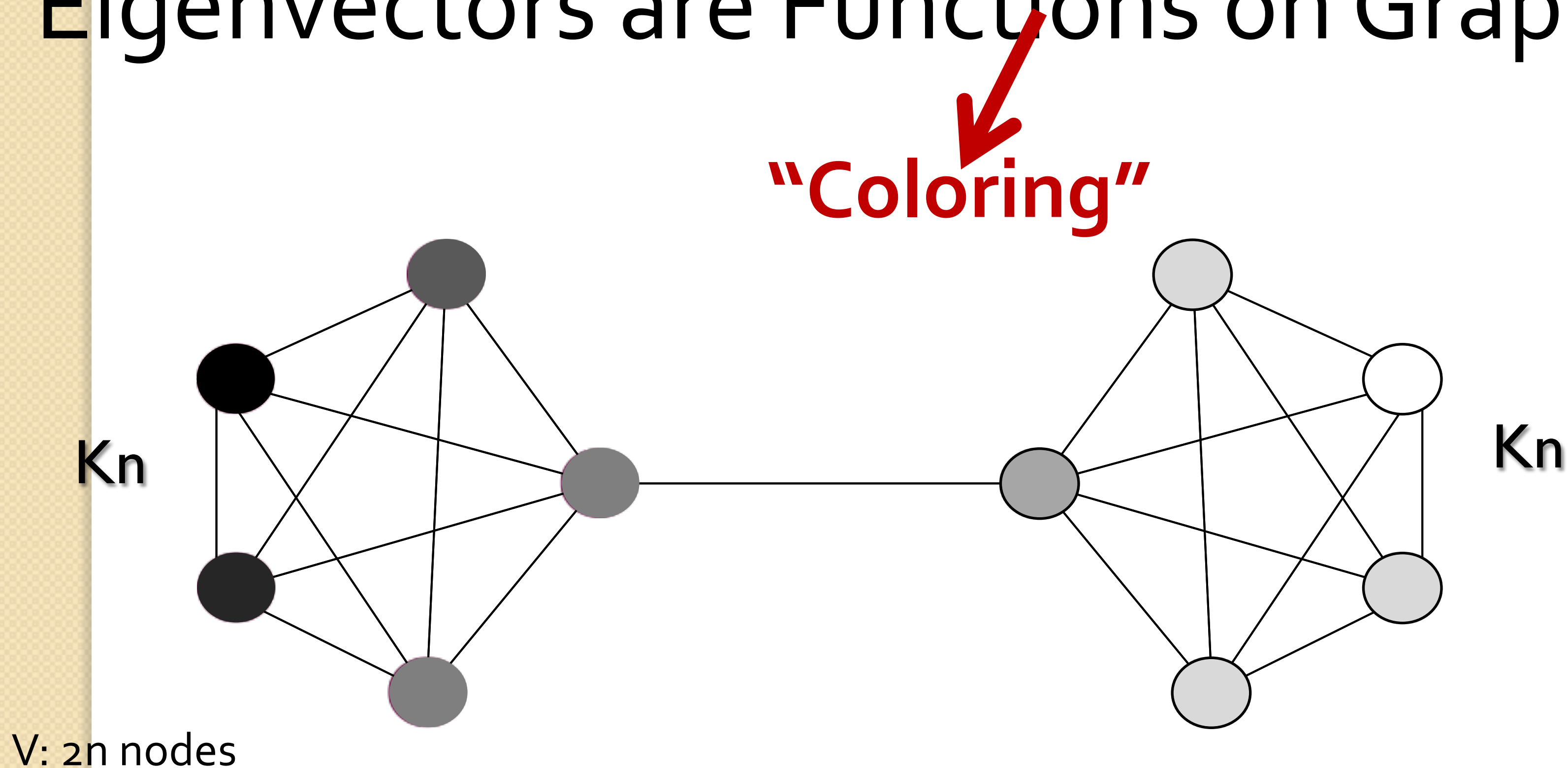
V: 2n nodes

$$Av = \mu v$$

$$v \in \Re^n, \quad v : V \to \Re$$

$v(i) =$ value at node i

Eigenvectors are Functions on Graph



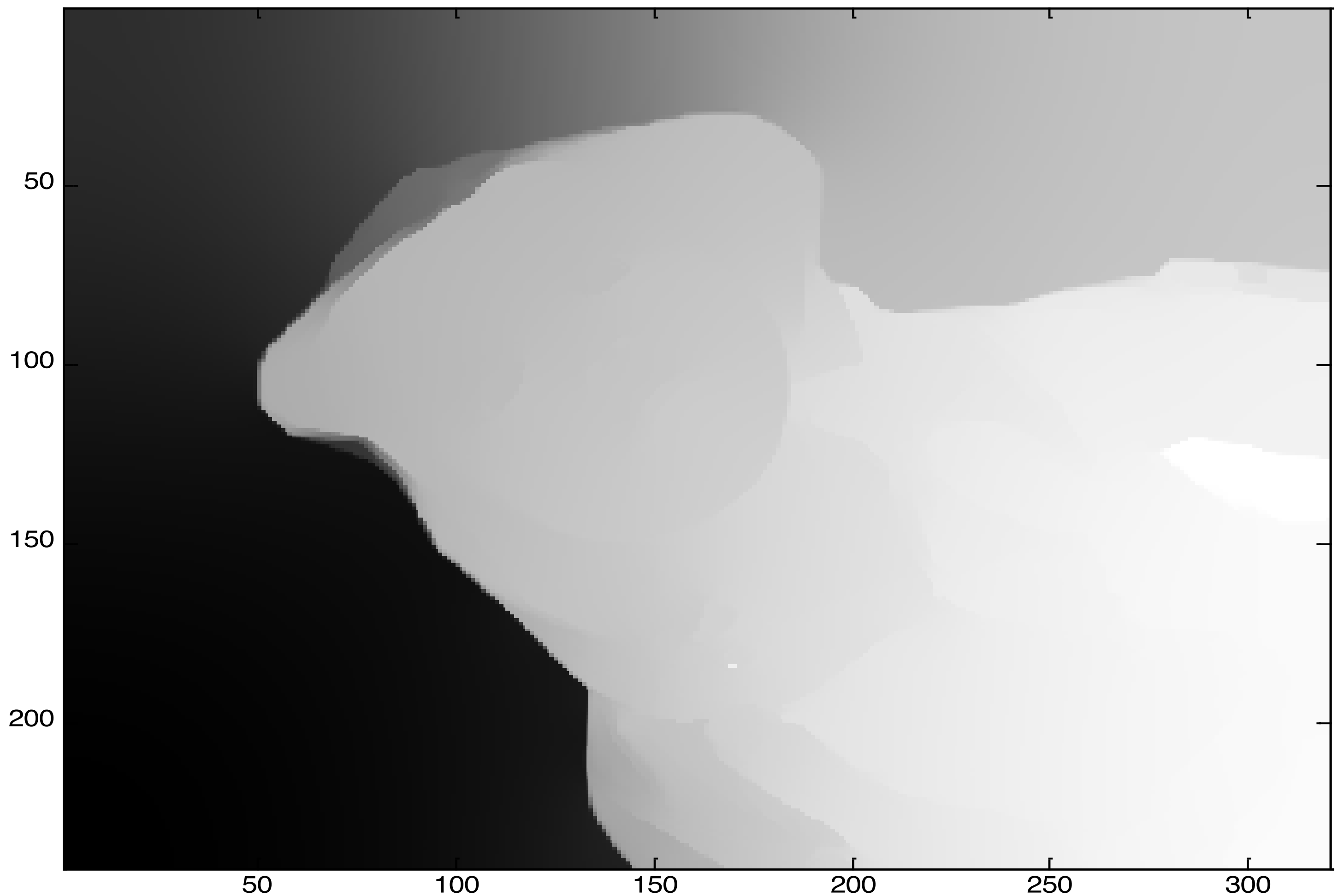
$$v \in \Re^n, \quad v : V \rightarrow \Re \qquad Av = \mu v$$

$v(i) =$ value at node i = different shade of grey

So, let's See the Eigenvectors

* Slides from Dan Spielman

The second eigenvector



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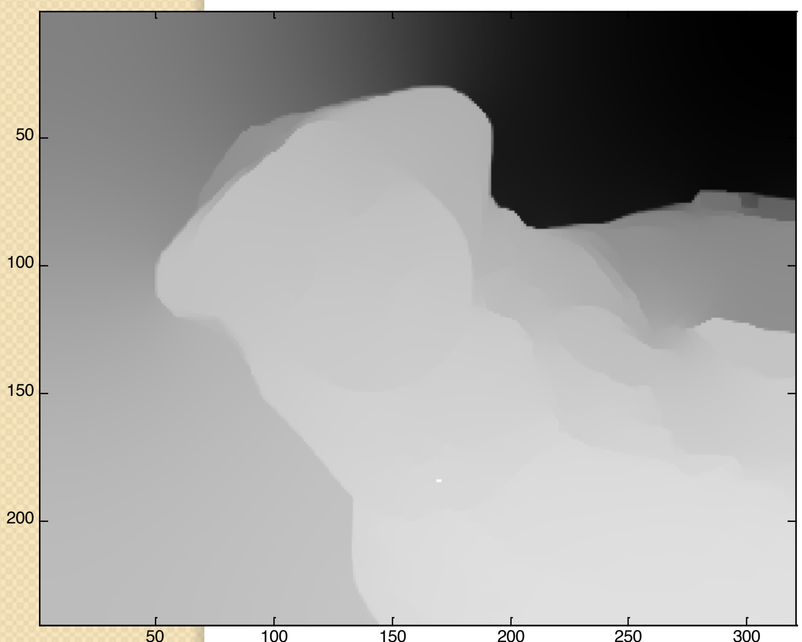
Third Eigenvector



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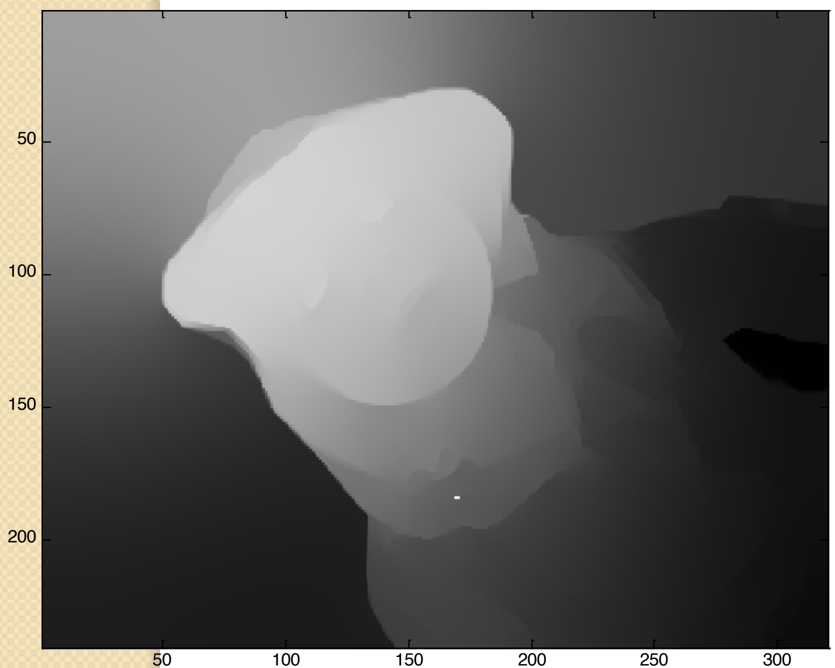
Fourth Eigenvector



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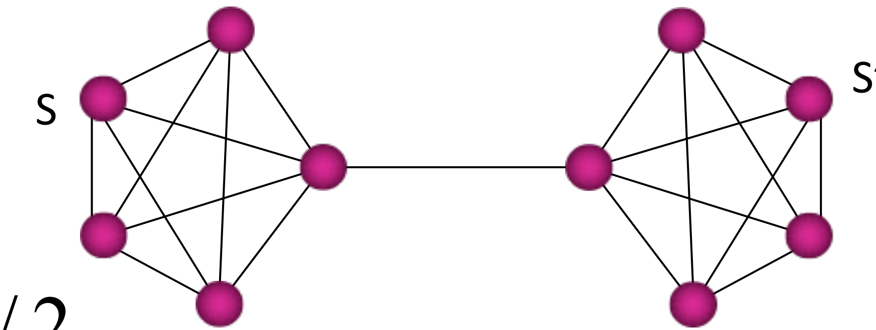
Cuts and Algebraic Connectivity

Cuts in a graph:



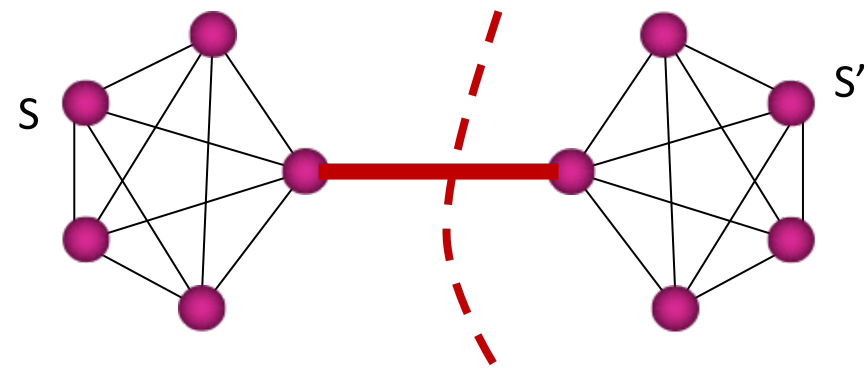
$$cut(S, S') = \frac{E(S, S')}{|S|}, |S| \leq n/2$$

Graph not well-connected when "easily" cut in two pieces

Cuts and eigenvalues

Edge-expansion:

$$h(G) = \min_{S:|S|\leq n/2} \frac{E(S,\bar{S})}{|S|}$$



Graph not well-connected when "easily" cut in two pieces

Would like to know Sparsest Cut but NP hard to find

How does algebraic connectivity relate to standard connectivity?

Theorem(Cheeger-Alon-Milman): $\lambda_2 \leq h(G) \leq \sqrt{2d_{\max}}\sqrt{\lambda_2}$

Today

- More on evectors and evalues.
- Evalues of d-regular graphs.
- Relation between eigenvalues and expansion (Cheeger, part 1).

A Remark on Notation

For convenience, we will often use the bra-ket notation for vecotrs:

- We denote vector $v = \begin{pmatrix} v_1 \\ \cdots \\ v_n \end{pmatrix}$ with a "bra": $|v\rangle$
- We denote the transpose vector $v^T = (v_1 \quad \cdots \quad v_n)$ with a "ket": $\langle v|$
- We denote the inner product $v^T u$ between two vectors v and u with a "braket": $\langle v|u\rangle = \langle v, u\rangle$

Evectors and Evalues

- Vector v is evector of matrix M with evalue λ if $Mv=\lambda v$.
- We are interested (almost always) in symmetric matrices, for which the following special properties hold:
  - If v_1, v_2 are evectors of A with evalues λ_1, λ_2 and $\lambda_1 \neq \lambda_2$, then v_1 is orthogonal to v_2. (Proof)
  - If v_1, v_2 are evectors of A with the same evalue λ, then v_1+v_2 is as well. The multiplicity of evalue λ is the dimension of the space of evectors with evalue λ.
  - Can assume that eigenvectors have unit length, since every multiple of an eigenvector is also an eigenvector.

Evectors and Evalues

- Generally, $Mv = \lambda v \Rightarrow (M - \lambda I)v = 0 \Rightarrow \det(M - \lambda I) = 0$.
- The determinant is an n-degree polynomial and has n roots, counting multiplicities.
- Every n-by-n symmetric matrix has n evalues $\{\lambda_1 \leq \cdots \leq \lambda_n\}$ counting multiplicities, and and orthonormal basis of corresponding evectors $\{v_1, \ldots, v_n\}$, so that $\mathrm{M}v_i = \lambda_i v_i$

- If we let V be the matrix whose i-th column is v_i, and D the diagonal matrix whose i-th diagonal is λ_i, we can compactly write MV=VD. Multiplying by V^T on the right, we obtain the eigendecomposition of M:

$$M = MV\ V^T = \mathrm{VD}\ V^T = \sum_i \lambda_i v_i v_i^T$$

Some eigenvalue theorems

- **Theorem 1**. Let $M \in R^{n\times n}$ symmetric. Then $\lambda_1 = \max_{x\in R^n, ||x||=1} \{x^T M x\}$, where $x^T M x = \sum_{i,j} x(i)x(j)M(i,j)$.

- Similarly, $\lambda_2 = \max_{x\in R^n, ||x||=1, x\perp x_1} \{x^T M x\}$
- $\max\{|\lambda_2|, \dots, |\lambda_n|\} = \max_{x\in R^n, ||x||=1} \{|x^T M x|\}$

Some eigenvalue theorems

- **Theorem 2**. Let G be a d-regular graph and M its adjacency matrix. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be its eigenvalues and $x_1, x_2, \dots, x_n$ the corresponding eigenvectors. Then $\lambda_1 = d$. Moreover, $x_1 = (1, \dots, 1)$.

Eigenvalues and connectivity

- **Theorem 2'**. Let G be a d-regular graph and M its adjacency matrix. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be its eigenvalues and $x_1, x_2, \dots, x_n$ the corresponding eigenvectors. Then $\lambda_1 = d$. If $\lambda_2 = d$ then the graph is disconnected. The converse is also true (ex). Alternatively, h(G)= 0 iff $\lambda_2 = d$.
- Generally, the more connected the graph is, the smaller λ_2 is.

Eigenvalues and expansion

- **Cheeger's Inequality**:

$$\frac{d-\lambda_2}{2} \leq h(G) \leq \sqrt{d(d-\lambda_2)}$$

- Both upper and lower bounds are tight (up to constant), as seen by path graph and complete binary tree.