Complexity Theory Lecture 12

Random Walks and Eigenvalues

Alexandra Kolla

Today

- Random walks on graphs review.
- Matrix form of random walks, lazy random walk.
- The stable distribution.
- Convergence and the second eigenvalue.
- Random walks on expanders.
- ST-UCONN in RL.

Random Walks on Graphs

- G=(V,E,w) weighted undirected graph.
- Random walk on G starts on some vertex and moves to a neighbor with prob. proportional to the weight of the corresponding edge.
- We are interested in the probability distribution over vertices after a certain number of steps.

Random Walks on Graphs

- G=(V,E,w) weighted undirected graph.
- Let vector $p_t \in R^n$ denote the probability distribution at time t. We will also write $p_t \in R^V$, and $p_t(u)$ for the value at vertex u.
- Since it's a probability vector, $p_t(u) \geq 0$ and $\sum_u p_t(u) = 1$ for every t.
- Usually, we start our walk at one vertex, so $p_0(u) = 1$ for some vertex u and 0 for the rest.

Random Walks on Graphs

- To derive p_t from p_{t+1} note that the probability of being at node u at time t+1 is the sum over all neighbors v of u of the probability that the walk was on v at time t times the probability it moved from v to u in one step:

$$p_{t+1}(u) = \sum_{v:(u,v)\in E} \frac{w(u,v)}{d(v)} p_t(v)$$

Where $d(v)$=$\sum_u w(u,v)$ is the weighted degree of v.

Lazy Random Walks

- We will often consider lazy random walks, which are a variant where we stay put with probability ½ at each time step, and walk to a random neighbor the other half of the time.

$$p_{t+1}(u) = \frac{1}{2} p_t(u) + \frac{1}{2} \sum_{v:(u,v)\in E} \frac{w(u,v)}{d(v)} p_t(v)$$

- Lazy random walks closely related to diffusion processes (at each time step, some substances diffuses out of each vertex)

Normalized Adjacency Matrix

- Need to define normalized version of Adjacency matrix.
- Normalized Adjacency matrix is what you would expect:

$$M_G = {D_G}^{-1/2} A_G {D_G}^{-1/2}$$

With eigenvalues $1 = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and first eigenvector $\sqrt{\mathbf{d}}$.

Normalized Adjacency Matrix

- We care about d-regular graphs.
- Normalized Adjacency matrix is what you would expect:

$$M_G = \frac{1}{d} A_G$$

With eigenvalues $1 = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$

and first eigenvector **1.**

Matrix Form of Random Walk

- Best way to understand random walks is with linear algebra. Equation

$$p_{t+1}(u) = \frac{1}{2} p_t(u) + \frac{1}{2} \sum_{v:(u,v)\in E} \frac{w(u,v)}{d} p_t(v)$$

Is equivalent to (verify)

$$p_{t+1} = \frac{1}{2}\left(I + \frac{1}{d} A\right) p_t$$

The lazy r.w. matrix is:

$$W_G = \frac{1}{2}(I + M) = \frac{1}{2}\left(I + \frac{1}{d} A_G\right)$$

Why Lazy Random Walks?

- $W_G = \frac{1}{2}(I + M) = \frac{1}{2}(I + \frac{1}{d} A_G)$
- All evals of W are between 1 and 0: Perron evalue of M is 1, so M has evalues between 1 and -1.
- We let $1 = \omega_1 \geq \omega_2 \geq \cdots \geq \omega_n \geq 0$
- Where $\omega_i = 1/2(1 + \mu_i)$ = $1/2(1 + \lambda_i/d)$

The Stable Distribution

$$W_G = \frac{1}{2}(I + M) = \frac{1}{2}(I + \frac{1}{d}A_G)$$

- Regardless of starting distribution, lazy r.w. always converges to stable distribution.
- In stable distribution, every vertex is visited with probability proportional to its (weighted) degree.

$$\boldsymbol{\pi}(\mathrm{i}) = \frac{\boldsymbol{d}(i)}{\sum_j \boldsymbol{d}(j)} = \frac{\mathbf{1}}{\mathbf{n}}$$

The Stable Distribution

- **π** is right evector of W with evalue 1.
- Other reason to consider lazy walks, is that they always converge. (e.g. consider bipartite graphs)
- Distribution converges to **π**. (Proof)

Rate of Convergeance

- Rate of convergence to the stable distribution is dictated by the second eigenvalue of W.
- Assume that r.w. starts at some vertex a. Let χ_a the characteristic vector of a, which is our starting distribution. For every vertex b, we will bound how far $p_t(\mathrm{b})$ can be from $\boldsymbol{\pi}(\mathrm{b})$.

Rate of Convergeance

- Assume that r.w. starts at some vertex a. Let χ_a the characteristic vector of a, which is our starting distribution. For every vertex b, we will bound how far $p_t(\mathrm{b})$ can be from $\boldsymbol{\pi}(\mathrm{b})$:

- **Theorem**. For all a,b, if $p_0 = \chi_a$ then
$$|p_t(\mathrm{b}) - \boldsymbol{\pi}(\mathrm{b})| \leq {\omega_2}^t$$

How Many Steps to Converge?

- To have $|p_t(\mathrm{b}) - \boldsymbol{\pi}(\mathrm{b})| \leq \varepsilon$, we need t to be such that ${\omega_2}^t \leq \varepsilon$.
- Define $\omega_2 = 1 - \gamma$, where γ is the spectral gap between first and second eigenvalue(remember discussion about expansion and large spectral gap).
- Number of steps to convergeance depends on 1/ γ , use $1 - \gamma \leq e^{-\gamma}$.

Mixing time for graphs

- Let's go back to thinking of non-lazy r.w. on d-regular, connected, non-bipartite graphs.
- It follows that $|p_t(\mathrm{b}) - \boldsymbol{\pi}(\mathrm{b})| \leq \frac{1}{2n}$ when $t \approx O\left(\frac{\log \mathrm{n}}{\gamma}\right) = O\left(\frac{\log \mathrm{n}}{1 - \frac{\lambda_2}{d}}\right)$ (mixing time)
- For expanders, $\gamma = \Omega(1)$. Set $\lambda = \frac{\lambda_2}{d}$.

Mixing time for graphs

- For any graph, we show $1 - \lambda = \gamma \geq \frac{1}{dn^2}$
- Use fact $(\sum_i |v_i|)^2 \leq n \sum_i v_i^2$

- Therefore, mixing time is $O(dn^2 \log n)$.

ST-UCONN and symmetric non-deterministic machines

- Undirected s,t, connectivity ST-UCONN: we are given undirected graph and the question is if there is path from s to t.
- Not known to be complete for NL, probably not, but complete for class SL (symmetric, non-deterministic TM with O(log n) space).

From previous lectures

- L ⊆SL ⊆ RL ⊆ NL.
- Reingold ‘04 showed in a breakthrough result that L=SL.
- We will see that ST-UCONN in RL in this lecture. (Aleliunas, Karp, Lipton, Lov´asz, Rackoff)
- Later on we will see Reingold’s theorem.