Computational Complexity. Lecture 19

Interactive Proofs.

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Today

- Adding randomness and interaction to NP.
- The class IP and its variants.
- IP for Graph non-Isomorphism.
- Private coins vs. public coins.

Characterization of NP, million-th time

- L is an NP language if there is a poly time algorithm V(.,.) and a polynomial p s.t.

$x \in L \Leftrightarrow$

$\exists y, |y| \leq p(|x|)\ and\ V(x,y)\ accepts$

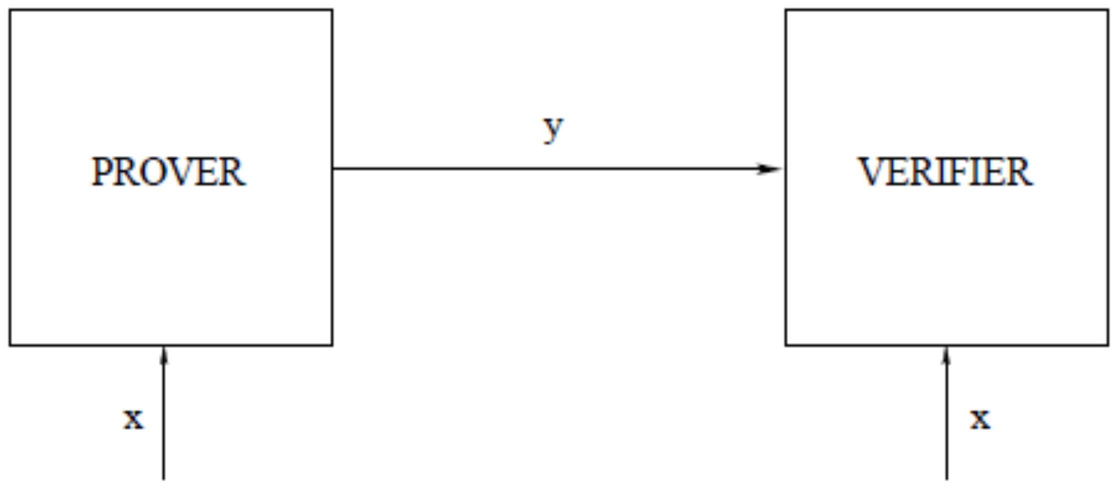
- Alternatively,

$x \in L \Rightarrow \exists y, |y| \leq p(|x|)\ and\ V(x,y)\ accepts$

$x \notin L \Rightarrow \forall y, |y| \leq p(|x|)\ V(x,y)\ rejects$

Completeness and soundness resp.

Prover /Verifier view of NP



Prover/verifier characterization of NP

- L is an NP language if there is a prover P and a poly time verifier (algorithm) V(.,.) p s.t.

$x \in L \Rightarrow$ P has strategy to convince V.

$x \notin L \Rightarrow$ P has no strategy to convince V.

- Strategy means the certificate of proof is polynomially small.
- Later will generalize to interaction where there is a sequence of messages exchanged and strategy means a function from the sequence of messages seen to the next message the prover sends.

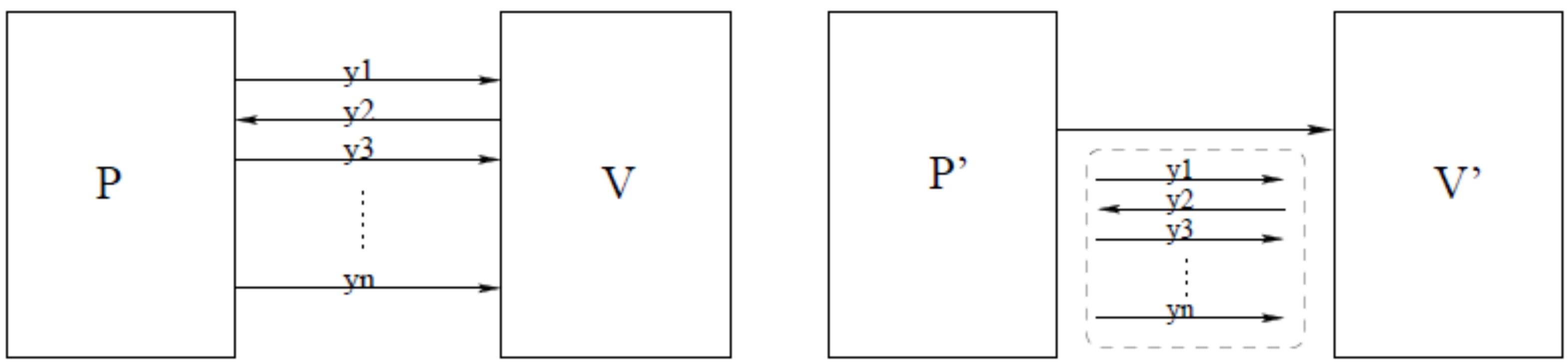
The class IP

- We will define the class IP with two more ingredients

- Randomness: V could be a randomized machine
- Interaction: unlike above where there is only one "round" of communication, verifier may ask several questions to prover based on the messages already seen.
- Both of the above are required.

NP + interaction

- **Theorem.** NP+interaction =NP



NP + randomness

- **Definition**. L is in MA if there exists a probabilistic polynomial time machine V such that:

$$x \in L \Rightarrow \exists y\ Pr[V(x,y)\ accepts] \geq \frac{2}{3}$$

$$x \notin L \Rightarrow \forall y\ \mathrm{Pr}[V(x,y)\ accepts] \leq \frac{1}{3}$$

- It is conjectured that MA=NP.
- It is known that if coNP $\subseteq$MA the polynomial hierarchy collapses.

- **Definition**. NP+randomness =MA

The class IP

- **Definition**. A language L is in IP(r(.)) iff there is a porbabilistic polynomial time verifier V such that:

x$\in$ L $\Rightarrow$

$\exists P\ Pr[V\ interacting\ with\ P\ accepts] \geq \frac{2}{3}$

x$\notin$ L $\Rightarrow$

$\forall P\ \mathrm{Pr}[Vinteracting\ with\ P\ accepts] \leq \frac{1}{3}$

V also uses at most r(|x|) rounds of interaction.

Public coins and the class AM

- **Definition**. A language L is in AM(r(.)) iff L is in IP(r(.)) and at each round the verifier sends a random message, that is a message that is completely random and independent of the previous communication.

Public coins vs. private coins

Theorem 1. $IP(r(n)) \subseteq AM(r(n)+2)$

Theorem 2. For all r >1, $AM(2r(n)) \subseteq AM(r(n))$

Corollary 3. $AM(O(1)) \subseteq AM(2)$

Theorem 4. $IP((O(1)) = AM((O(1)) = AM(2)$

Public coins vs. private coins

Theorem 5. IP(poly(n)) =PSPACE (next time).

Theorem 6. If co NP$\subseteq IP(O(1))$ then the polynomial hierarchy collapses.

IP for Graph non-Isomorphism

- We will next see and IP with constant number of rounds for GNI:
- By previous results, it is also in AM(2).
- **Theorem**. GNI $\in$AM(2).
- We show that next from scratch. Similar proof goes for theorem 4.

IP for Graph non-Isomorphism

- **Theorem**. If GI is NP-complete then the polynomial hierarchy collapses (to the second level).