Computational Complexity. Lecture 3

Boolean Circuits

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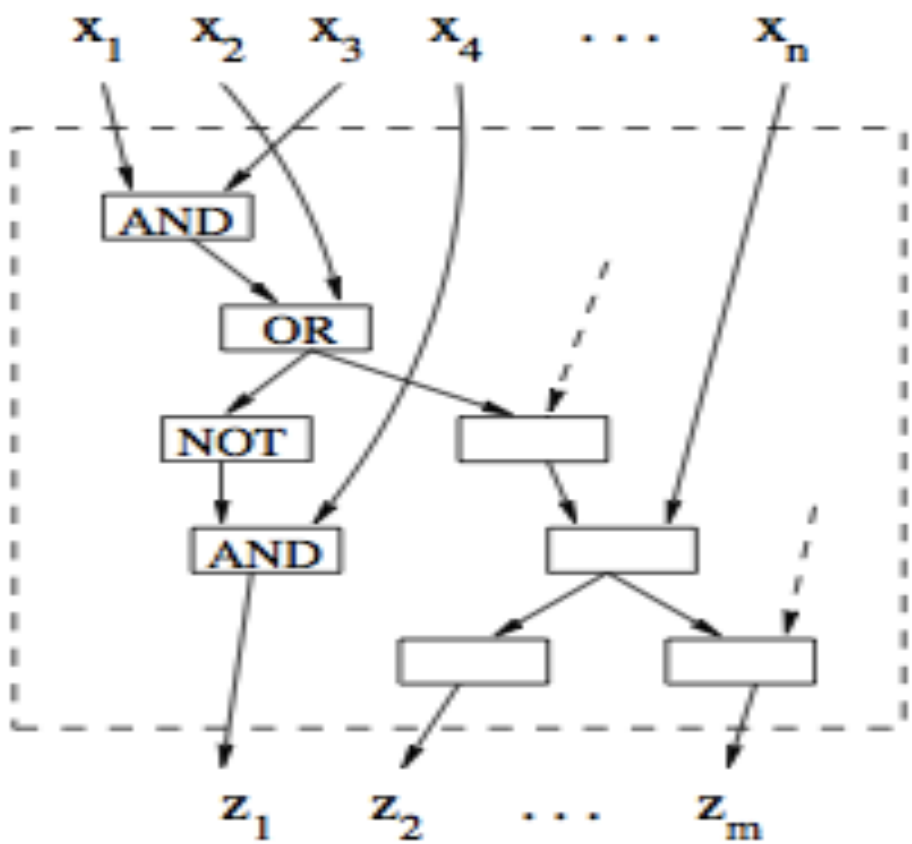
Today

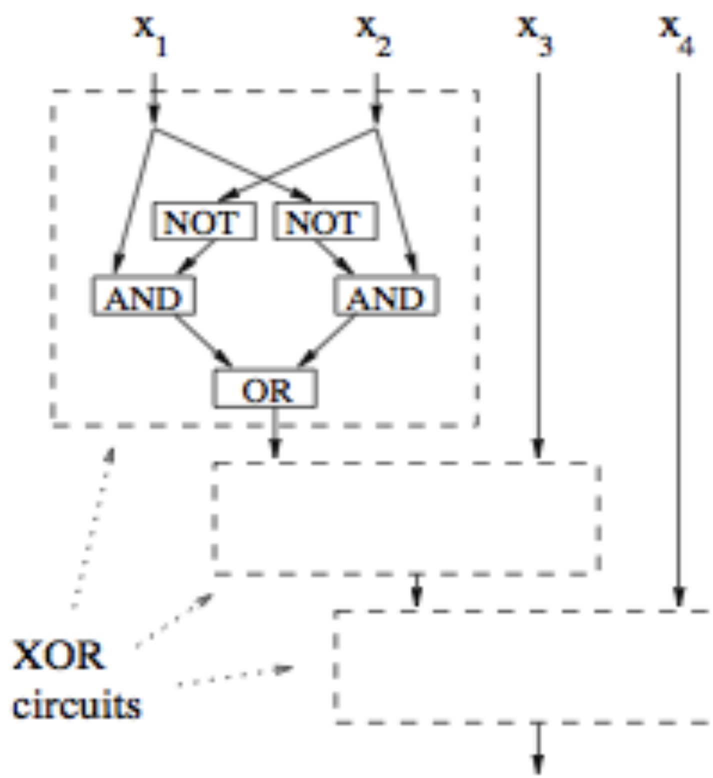
- Boolean circuits
- Poly size circuits can simulate poly computations
- Relations between complexity classes
- Karp-Lipton

Circuits

- Circuit C has n inputs, m outputs and is constructed with AND, OR, NOT gates.
- Each gate has in-degree 2 except the NOT gate which has in-degree 1
- Circuit C computes function $f_C: \{0,1\}^n \rightarrow \{0,1\}^m$
- SIZE(C)=number of AND and OR gates (we don't count NOT gates)

Circuits





A circuit computing the boolean function $f_C(x_1 x_2 x_3 x_4) = x_1 \oplus x_2 \oplus x_3 \oplus x_4$

Circuits

- To be compatible with other complexity classes, need to extend the model to arbitrary input sizes:
- **Definition 1**. Language L is solved by a family of circuits $\{C_1, C_2, \ldots, C_n, \ldots\}$ if for every $n \geq 1$ and for every x s.t. $|x| = n$

$$x \in L \Leftrightarrow f_{C_n}(x) = 1$$

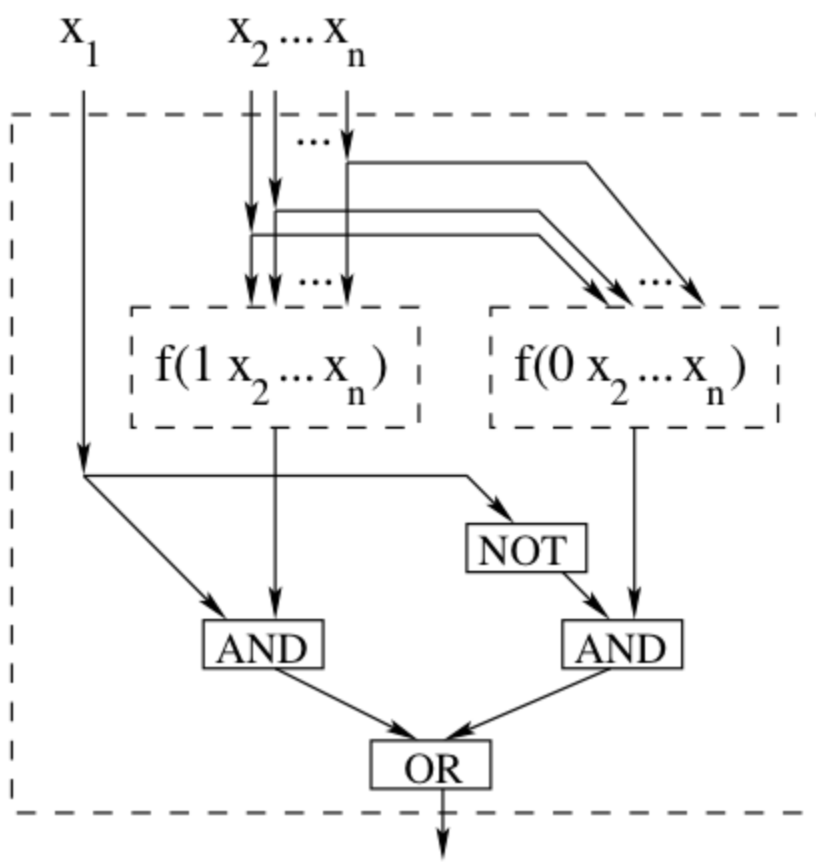
- **Definition 2**. Language L $\in \mathrm{SIZE}(s(n))$ if L is solved by a family of circuits $\{C_1, C_2, \ldots, C_n, \ldots\}$ where C_i has at most s(i) gates.

Relation to other complexity classes

- Unlike other complexity classes where there are languages of arbitrarily high complexity, the size complexity of a problem is always at most exponential
- **Theorem**. For every language L,

$$L \in \mathrm{SIZE}(O(2^n))$$

Relation to other complexity classes



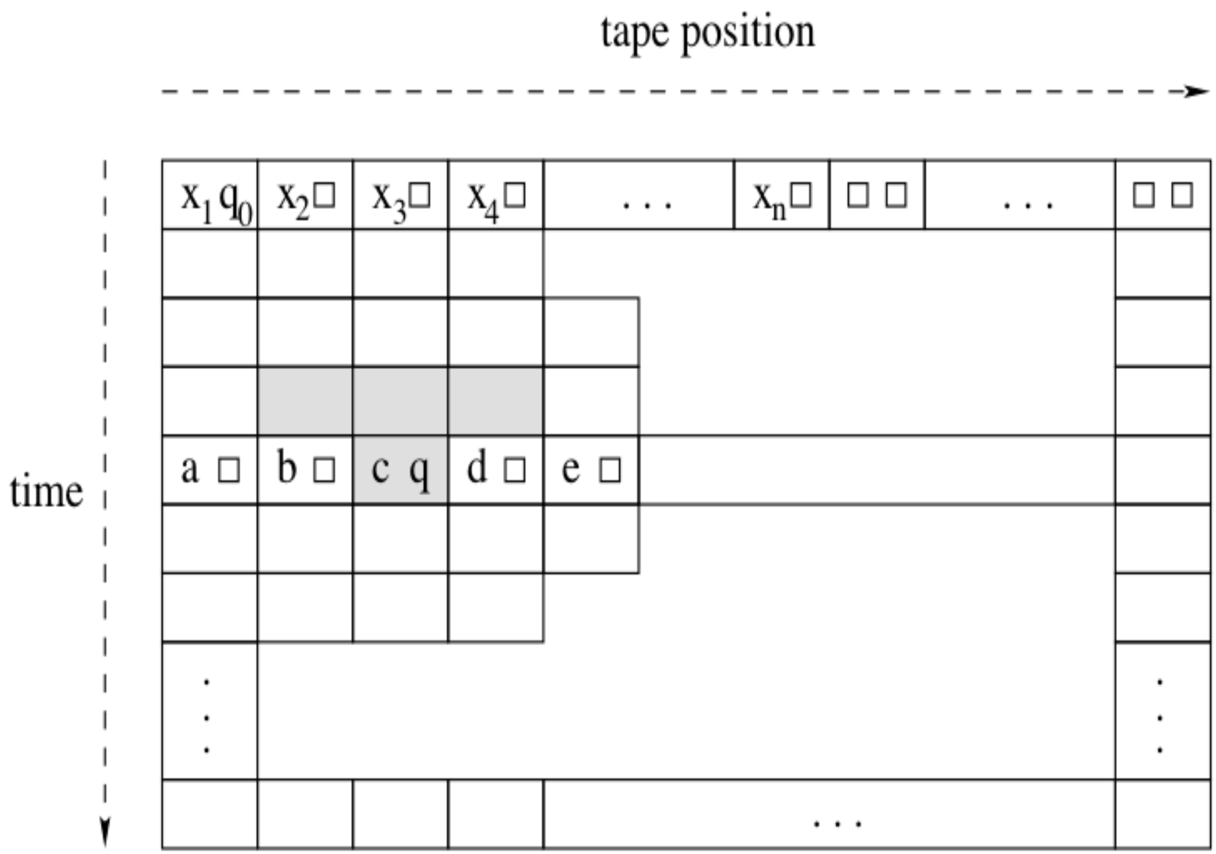
Relation to other complexity classes

- Exponential bound is nearly tight
- **Theorem**. There are languages L such that L $\notin \mathrm{SIZE}(2^{o(n)})$. In particular, for every n$\geq$11, there exists $f \quad : \{0,1\}^n \rightarrow \{0,1\}$ that cannot be computed by a circuit of size $2^{o(n)}$.

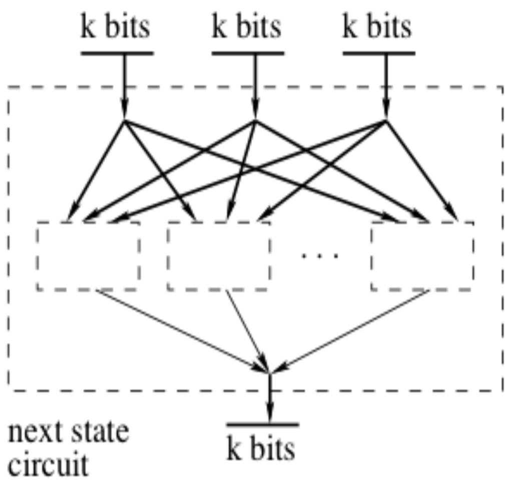
Relation to other complexity classes

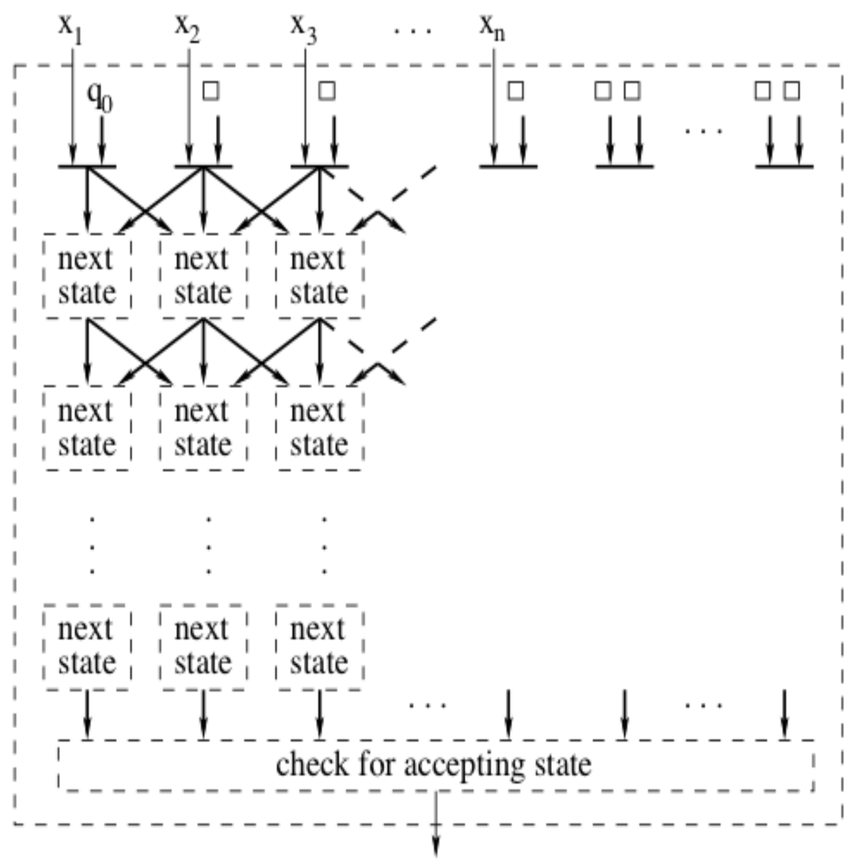
- Efficient computations can be simulated by small circuits
- **Theorem**. If $\mathrm{L} \in \mathrm{DTIME}(\mathrm{t}(\mathrm{n}))$, then $\mathrm{L} \in \mathrm{SIZE}(\mathrm{O}(t^2(n)))$

Relation to other complexity classes



Relation to other complexity classes





Relation to other complexity classes

- Efficient computations can be simulated by small circuits
- **Theorem**. If $\mathrm{L} \in \mathrm{DTIME}(\mathrm{t}(\mathrm{n}))$, then $\mathrm{L} \in \mathrm{SIZE}(\mathrm{O}(t^2(n)))$
- **Corollary**. P $\subseteq \mathrm{SIZE}(n^{O(1)})$
- However, P $\neq \mathrm{SIZE}(n^{O(1)})$. In fact, there are undecidable languages in $\mathrm{SIZE}(\mathrm{O}(1))$ (ex)

Karp-Lipton-Sipser

- **Theorem**. If NP $\subseteq \mathrm{SIZE}(n^{O(1)})$ then $\mathrm{PH} = \Sigma_2$

Karp-Lipton-Sipser

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