Linear Algebra

CSCI 2820

Lecture 12

Prof. Alexandra Kolla

Alexandra.Kolla@Colorado.edu
ECES 122

Today

- More on Projections
- Gram-Schmidt re-explained

Orthogonal Projections

$$\boxed{\hat{\mathbf{y}} = \text{proj}_L\, \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}}$$

Eg. Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

orthogonal projection of y onto u:

$$\hat{y} = \text{proj}_u \vec{y} = \frac{\langle y,u \rangle}{\|u\|^2} \cdot \vec{u} = \frac{7\cdot 4 + 6\cdot 2}{4^2+2^2}\vec{u} = \frac{40}{20}\vec{u} = 2\vec{u} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$z = y - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

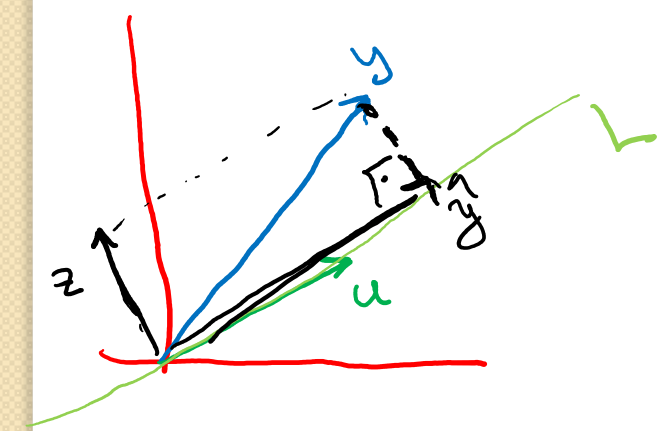
- sanity check 1: $y = \hat{y} + z = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ ✓
- sanity check 2: $\hat{y}, z$ are orthogonal

$$\langle \hat{y}, z \rangle = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -8+8 = 0$$ ✓

Orthogonal Projections

$$\boxed{\hat{\mathbf{y}} = \text{proj}_L\, \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}}$$

Eg. Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$



what is distance from some vector $\vec{y}$ to a line L (defined by a vector $\vec{u}$).

(step1) calculate $\hat{y} = \text{proj}_L \vec{y}$

(step2) $\hat{y}$ is closest point

$\|\vec{y} - \hat{y}\|$ = distance from $\vec{y}$ to L

Orthogonal sets

Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

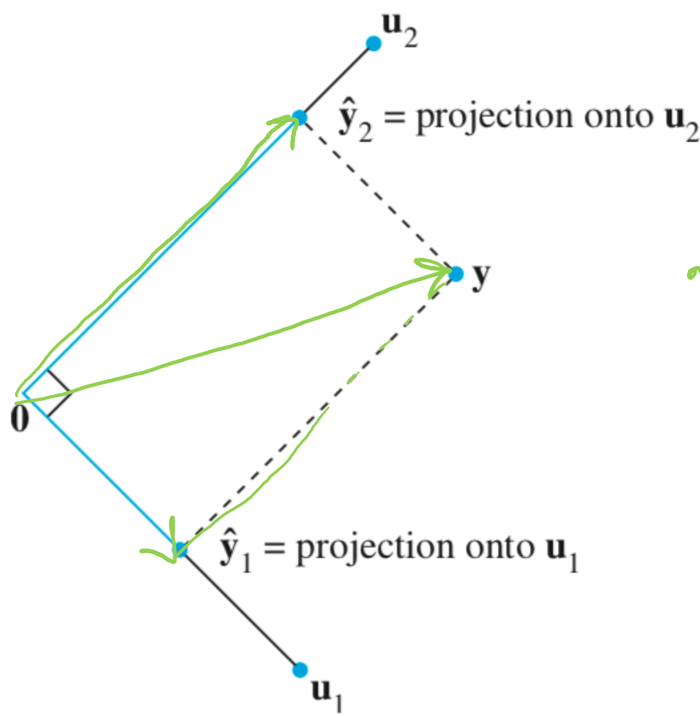
proj of $\vec{y}$ onto span$\{\vec{u_p}\}$

$$\mathbf{y} = \underbrace{c_1\mathbf{u}_1}_{\text{proj of } \vec{y} \text{ onto span}\{\vec{u_1}\}} + \cdots + c_p\mathbf{u}_p$$

are given by

$$\boxed{c_j = \frac{\mathbf{y}\cdot\mathbf{u}_j}{\mathbf{u}_j\cdot\mathbf{u}_j}} \qquad (j = 1, \dots, p)$$

Orthogonal Projections



$$\vec{y} = \frac{\langle y, u_1 \rangle}{\|u_1\|^2} \vec{u_1} + \frac{\langle y, u_2 \rangle}{\|u_2\|^2} \cdot \vec{u_2}$$

Orthogonality

True or False?

a. Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.

b. If a set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ has the property that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$, then S is an orthonormal set.

Orthogonality

True or False?

The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

Orthogonality

Given $\mathbf{u} \neq \mathbf{0}$ in $\mathbb{R}^n$, let $L = \text{Span}\{\mathbf{u}\}$. Show that the mapping $\mathbf{x} \mapsto \text{proj}_L\, \mathbf{x}$ is a linear transformation.

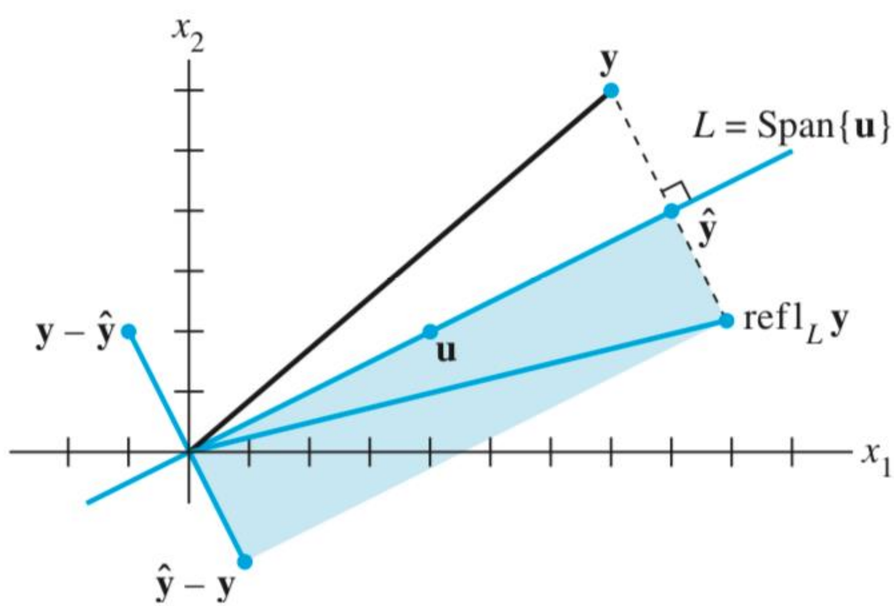
$f(\vec{x}) = \text{proj}_L \vec{x}$ $\quad f: \mathbb{R}^n \to \mathbb{R}^n$

Orthogonality

Given $\mathbf{u} \neq \mathbf{0}$ in $\mathbb{R}^n$, let $L = \text{Span}\{\mathbf{u}\}$. For $\mathbf{y}$ in $\mathbb{R}^n$, the **reflection of y in** L is the point $\text{refl}_L\, \mathbf{y}$ defined by

$$\text{refl}_L\, \mathbf{y} = 2 \cdot \text{proj}_L\, \mathbf{y} - \mathbf{y}$$

See the figure, which shows that $\text{refl}_L\, \mathbf{y}$ is the sum of $\hat{\mathbf{y}} = \text{proj}_L\, \mathbf{y}$ and $\hat{\mathbf{y}} - \mathbf{y}$. Show that the mapping $\mathbf{y} \mapsto \text{refl}_L\, \mathbf{y}$ is a linear transformation.



The reflection of **y** in a line through the origin.

Orthogonal Projections

$\vec{y}$, subspace W of $\mathbb{R}^n$, there is a vector $\hat{y}$ in W such that

(1) $\hat{y}$ is the unique vector in W for which $y-\hat{y}$ is orthogonal to W (we called z before)

(2) $\hat{y}$ is the unique vector in W closest to $\vec{y}$.

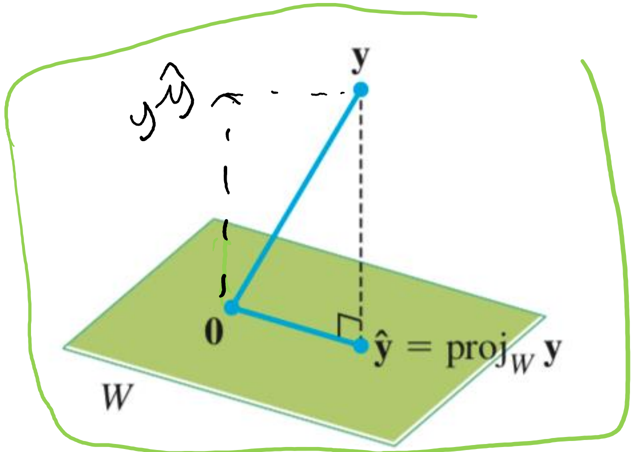
$$\vec{y} = \underbrace{c_1\vec{u}_1 + c_2\vec{u}_2}_{\vec{z}_1} + \underbrace{\cdots + c_n\vec{u}_n}_{\vec{z}_2}$$

$$\vec{y} = \vec{z}_1 + \vec{z}_2$$

$$\vec{z}_1 = \text{proj}_{\text{span}\{\vec{u}_1, \vec{u}_2\}} \vec{y}$$

$$\vec{z}_2 = \text{proj}_{\text{span}\{\vec{u}_3, \ldots, \vec{u}_n\}} \vec{y}$$

Orthogonal Projections



$\hat{y}$ = orthogonal proj of $\vec{y}$ on W

The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \tag{1}$$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$. In fact, if $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is any orthogonal basis of W, then

$$\boxed{\hat{\mathbf{y}} = \underbrace{\frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1}_{\text{proj}_{\text{span}\{\vec{u}_1\}}\vec{y}} + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p} \tag{2}$$

$\mathbf{u}_1\cdot\mathbf{u}_1 = 1$, $\mathbf{u}_p\cdot\mathbf{u}_p = 1$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

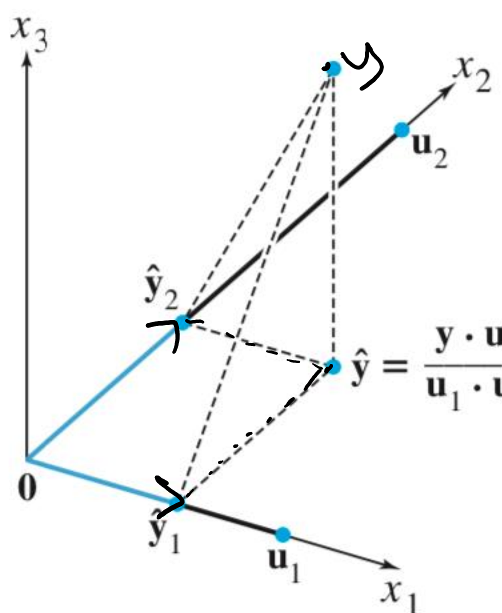
if $\{\vec{u}_1, \dots, \vec{u}_p\}$ orthonormal: $\hat{y} = \langle y, u_1\rangle \vec{u}_1 + \dots + \langle y, u_p\rangle \vec{u}_p$

since $\langle u_i, u_i\rangle = \|u_i\|^2 = 1$

Orthogonal Projections

Orthogonal Projections

- W is 1-dim ✓
- W is 2 or more dim



$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \hat{\mathbf{y}}_1 + \hat{\mathbf{y}}_2$$

$W = \text{span}\{\vec{x}_1, \vec{x}_2\}$

$W^\perp = \text{span}\{\vec{x}_3\}$

Orthogonal Projections

eg

$\vec{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$W = \text{span}\{\vec{u}_1, \vec{u}_2\}$

distance of $\vec{y}$ to W?

- step 1: calculate proj of $\vec{y}$ onto W, $\hat{y}$
- step 2: $\|y - \hat{y}\|$

- check $\{\vec{u}_1, \vec{u}_2\}$ is orthogonal basis : $\langle \vec{u}_1, \vec{u}_2 \rangle = 0$ ✓

$$\hat{y} = \frac{\langle \vec{y}, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2}\vec{u}_1 + \frac{\langle \vec{y}, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2}\vec{u}_2$$

$$= \frac{15}{30}\vec{u}_1 + \frac{-21}{6}\vec{u}_2 = \frac{1}{2}\vec{u}_1 - \frac{7}{2}\vec{u}_2 = \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}$$

$$y - \hat{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix} - \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

$\|y - \hat{y}\|^2 = 3^2 + 6^2 = 45$

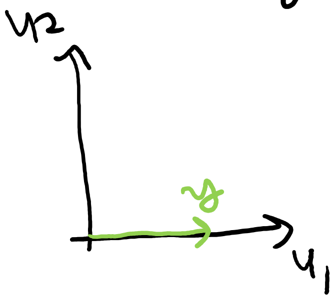
$\|y - \hat{y}\| = \sqrt{45}$ dist. of $\vec{y}$ to W

Orthogonal Projections

Properties of orthogonal projections

- $\{\vec{u}_1, \dots, \vec{u}_p\}$ orthogonal basis for W

  if $\vec{y}$ is already in W, $\text{proj}_W \vec{y} = \vec{y}$



$W = \text{Span}\{\vec{u}_1\}$

Thm: Best approximation theorem

Let W be a subspace of $\mathbb{R}^n$. $\vec{y} \in W$, $\hat{y}$ the orthogonal projection of $\vec{y}$ onto W. Then $\hat{y}$ is the closest point in W to $\vec{y}$.

$\boxed{\|y - \hat{y}\| < \|y - v\|}$ for any $v \neq \hat{y}$, $v \in W$.

Gram Schmidt.

Algorithm 5.1 Gram–Schmidt algorithm

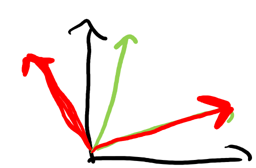
given n-vectors $a_1, \ldots, a_k$

for $i = 1, \ldots, k$,

1. *Orthogonalization.* $\tilde{q}_i = a_i - (q_1^T a_i)q_1 - \cdots - (q_{i-1}^T a_i)q_{i-1}$
2. *Test for linear dependence.* if $\tilde{q}_i = 0$, quit.
3. *Normalization.* $q_i = \tilde{q}_i / \|\tilde{q}_i\|$ $\qquad \|q_i\| = 1$

$\tilde{q}_1 = \vec{a}_1$ ✓

$\tilde{q}_2 = \vec{a}_2 - \overbrace{\langle q_1, a_2 \rangle \vec{q}_1}^{\text{proj of } \vec{a}_2 \text{ onto } \vec{q}_1}$



$\tilde{q}_3 = \vec{a}_3 - \boxed{\underbrace{\langle q_1, a_3 \rangle \vec{q}_1}_{\text{proj of } \vec{a}_3 \text{ ont } \vec{q}_1} - \underbrace{\langle q_2, a_3 \rangle \vec{q}_2}_{\text{proj of } \vec{a}_3 \text{ onto } \vec{q}_2}} \longrightarrow$ proj onto subspace $\text{span}\{\vec{q}_1, \vec{q}_2\}$

$\vdots$

Gram Schmidt

Gram Schmidt