Linear Algebra

CSCI 2820

Lecture 14

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ECES 122

Exercises

Norm of matrix-vector product. Suppose A is an $m \times n$ matrix and x is an n-vector. A famous inequality relates $\|x\|$, $\|A\|$, and $\|Ax\|$:

$$\|Ax\| \leq \|A\|\|x\|.$$

The left-hand side is the (vector) norm of the matrix-vector product; the right-hand side is the (scalar) product of the matrix and vector norms. Show this inequality. *Hints.* Let a_i^T be the ith row of A. Use the Cauchy–Schwarz inequality to get $(a_i^T x)^2 \leq \|a_i\|^2 \|x\|^2$. Then add the resulting m inequalities.

Exercises

Today

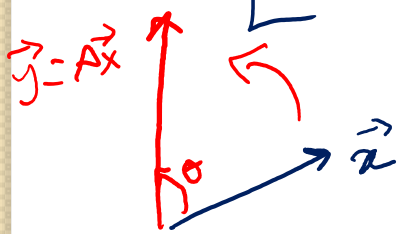
- Matrix Examples
- Matrix Operations
- Examples and exercises

Matrix examples: Rotation

2x2 case : Rotation by θ

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

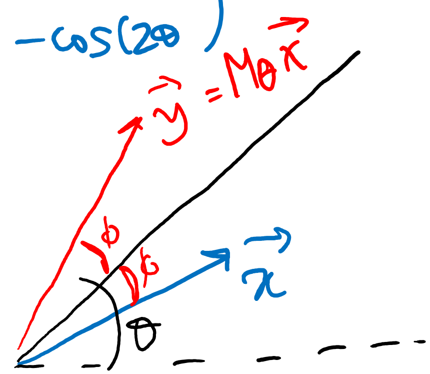
$\vec{x}$ then $\vec{y} = R_\theta \vec{x}$



Reflection matrix (by θ)

$$M_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

$\vec{y} = M_\theta \vec{x}$



Matrix examples: Selector/Permutation

$m \times n$ selector matrix $A = \begin{bmatrix} \vec{e}_{k_1}^{\,T} \\ \vdots \\ \vec{e}_{k_m}^{\,T} \end{bmatrix}$ $k_i \in [n]$

$\underline{\vec{y} = A\vec{x}} = \underline{(x_{k_1}, \ldots, x_{k_m})}$

special case? I : $\vec{y} = I\vec{x} = \vec{x}$

<u>Permutation matrices:</u> (A)

$n \times n$: A, A^T both selectors

$\vec{y} = A\vec{x} = (x_{\pi_1} \ldots x_{\pi_n})$

$\pi = (3, 1, 2)$: $A_\pi = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$A_\pi \vec{x} = (x_3, x_1, x_2)$

Matrix examples: Incidence

Directed graph, $V = \{1, \dots, n\}$, directed edges $\{1, \dots, m\}$

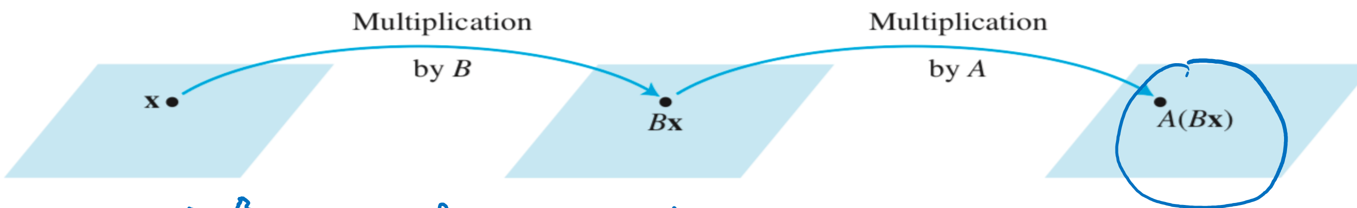
$$i \xrightarrow{e} j$$

Incidence matrix $n \times m$

$$A = \begin{array}{c} \\ 1 \\ \vdots \\ i \\ \vdots \\ n \end{array} \begin{array}{c} 1 \quad \cdots \quad m \\ \left[\begin{array}{ccc} & & \\ & \bigcirc & \\ & & \\ & & \end{array} \right] \end{array}$$

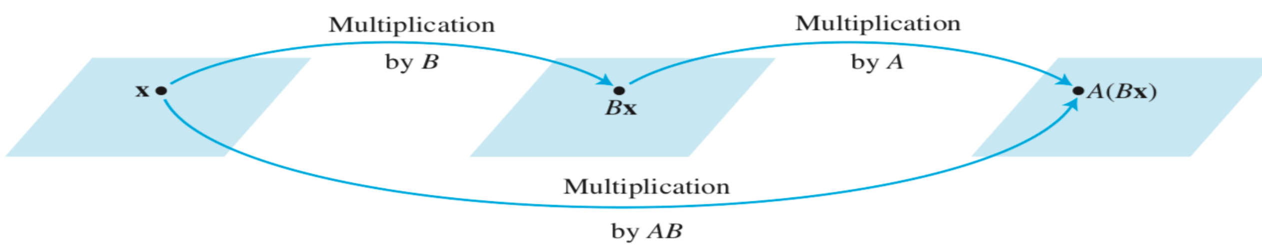
$$a_{ie} = \begin{cases} 1 & \text{if edge } e \text{ points to node } i \\ -1 & \text{if edge } e \text{ points from node } i \\ 0 & \text{o.w.} \end{cases}$$

Matrix multiplication



so far: $\vec{x} \xrightarrow{B} \underbrace{B\vec{x}}_{=\,\vec{y}} \xrightarrow{A} A(B\vec{x}) = A\vec{y}$

$$\vec{x} \xrightarrow{AB} A(B\vec{x})$$ (multiply $\vec{x}$ with $A \cdot B$)



Matrix multiplication

DEFINITION

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$. That is,

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$$

$$B = \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_p \end{bmatrix}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$$

$$\vec{y} = B\vec{x} = x_1\vec{b}_1 + \cdots + x_p\vec{b}_p$$

$$A\vec{y} = A(B\vec{x}) = A(x_1\vec{b}_1) + \cdots + A(x_p\vec{b}_p)$$

$$= x_1(A\vec{b}_1) + \cdots + x_p(A\vec{b}_p)$$

$$= \underbrace{\begin{bmatrix} A\vec{b}_1 & \cdots & A\vec{b}_p \end{bmatrix}}_{AB}\vec{x}$$

Matrix multiplication

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$A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ (2×2), $\quad B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$ (2×3)

$A \cdot B = ?$

$A\vec{b}_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$

$A\vec{b}_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix}$

$A\vec{b}_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$

$AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & A\vec{b}_3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$

Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B. (shown last time for $A\vec{x}$)

AB : same # rows as A (m rows), same # colums as B (n col.)

Matrix multiplication

Matrix multiplication

ROW–COLUMN RULE FOR COMPUTING *AB*

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$= \langle \overrightarrow{row_i}(A), \overrightarrow{col_j}(B) \rangle$

eg. $AB = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$

$\underset{1 \times m}{\overrightarrow{row_i}(AB)} = \overrightarrow{row_i}(A) \cdot \underset{m \times n}{B}$

$(AB)_{13} = 2 \cdot 6 + 3 \cdot 3 = 12 + 9 = 21$

$(AB)_{21} = 1 \cdot 4 - 5 \cdot 1 = 4 - 5 = -1$

Matrix multiplication

Properties of matrix mult:

A $m \times n$ matrix, B, C have correct sizes

a. $A(BC) = (AB)C$ associative $\rightsquigarrow ABC$

b. $A(B+C) = AB + AC$ left distributive

c. $(B+C)A = BA + CA$ right distr

d. $r(AB) = (rA)B = A(rB)$ $\forall\, r \in \mathbb{R}$

e. $I_m A = A = A I_n$ (identity for matrix mult)

order matters! Generally, $AB \neq BA$

if $AB = BA$ then we say A and B commute

Matrix ~~multiplication~~ powers

A $\boxed{n \times n}$ matrix, $k \in \mathbb{N}_+$

$$A^k = \underbrace{A \cdot A \cdots A}_{k}$$

Q: what if A is $n \times m$?
$m \neq n$.

$$A_{n \times m} \cdot A_{n \times m}$$

convention: $A^0 = I$

$(x^0 = 1)$

Matrix ~~Powers~~ Mult.

$$A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

do they commute? $(AB \stackrel{?}{=} BA)$

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}$$

$AB \neq BA$, A, B do not commute!

To remember:

① In general, $AB \neq BA$

② cancelation laws don't hold for matrix mult: if $AB = AC$, then we cannot deduce that $B = C$.

③ if $AB = \emptyset$ we cannot conclude $A = \emptyset$ or $B = \emptyset$

Practice Problems

Since vectors in $\mathbb{R}^n$ may be regarded as $n \times 1$ matrices, the properties of transposes in Theorem 3 apply to vectors, too. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Compute $(A\mathbf{x})^T$, $\mathbf{x}^T A^T$, $\mathbf{x}\mathbf{x}^T$, and $\mathbf{x}^T\mathbf{x}$. Is $A^T\mathbf{x}^T$ defined? NOT $\underset{2\times 2}{A^T}$ $\underset{1\times 2}{x^T}$

- $A\vec{x} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, $(A\vec{x})^T = [-4, 2]$
- $x^T A^T = \underset{1\times 2}{[5 \;\; 3]} \underset{2\times 2}{\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}} = [-4 \;\; 2]$ $\quad (A\vec{x})^T = x^T A^T$

general rule: $(A \cdot B)^T = B^T A^T$

- $\vec{x}\vec{x}^T = \underset{2\times 1}{\begin{bmatrix} 5 \\ 3 \end{bmatrix}} \underset{1\times 2}{[5 \;\; 3]} = \underset{2\times 2}{\begin{bmatrix} 25 & 15 \\ 15 & 9 \end{bmatrix}}$ ("outer product")
- $x^T x = \underset{1\times 2}{[5 \;\; 3]} \underset{2\times 1}{\begin{bmatrix} 5 \\ 3 \end{bmatrix}} = \underset{1\times 1}{[25+9]} = 34$ ("inner product")

Practice Problems

Suppose A is an $m \times n$ matrix, all of whose rows are identical. Suppose B is an $n \times p$ matrix, all of whose columns are identical. What can be said about the entries in AB?

- $AB = [Ab_1 \; \cdots \; Ab_n] = [Ab_1 \; \cdots \; Ab_1]$

- $\vec{row}_i(AB) = \vec{row}_i(A) \cdot B$ $\Rightarrow$ all rows of AB same

$\Rightarrow$ all entries of AB identical.

Practice Problems

Show that if the columns of B are linearly dependent, then so are the columns of AB.

$B = [\vec{b}_1 \ \ldots \ \vec{b}_n]$

$c_1\vec{b}_1 + \cdots + c_n\vec{b}_n = 0$ (*) c_i not all zero

$AB = [A\vec{b}_1 \ \ldots \ A\vec{b}_n]$. need to show that

there exist scalars d_i, not all zero s.t.:

$\underline{d_1 A\vec{b}_1} + \cdots + d_n A\vec{b}_n = 0$

$\Rightarrow A(d_1\vec{b}_1) + \cdots + A(d_n\vec{b}_n)$

$\Rightarrow A(d_1\vec{b}_1 + \cdots + d_n\vec{b}_n)$, take $\boxed{d_i = c_i}$ $\forall i$

$\Rightarrow A(\underbrace{c_1\vec{b}_1 + \cdots + c_n\vec{b}_n}) = A\vec{0} = 0$

Matrix inverse

we have a multiplicative inverse for all $r \neq 0$, $r \in \mathbb{R}$

e.g. $r = 5$, $5^{-1} = 1/5$: $\underbrace{5^{-1} \cdot 5}_{} = \underbrace{5 \cdot 5^{-1}}_{} = 1$

- an $n \times n$ matrix A is <u>invertible</u> if there is an $n \times n$ matrix C such that

$$C \cdot A = I \quad \text{and} \quad A \cdot C = I.$$

$\boxed{\text{C inverse of A}}$

- claim : C is unique

<u>proof</u>: Assume there was another one, B

then $B = B \cdot I = B(AC) = (\underbrace{BA}_{})C$

$= I \cdot C = C$

inverse A^{-1}, $AA^{-1} = A^{-1}A = I$

- A matrix is not invertible we call it singular

Inverse of 2×2 matrix

Thm: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad-bc \neq 0$

then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(easy to check!)

$$\boxed{\det A = ad-bc}$$

Thm says: a 2×2 matrix is invertible iff $\det A \neq 0$

- if A is $n\times n$ invertible, then for each $\vec{b} \in \mathbb{R}^n$: $A\vec{x} = \vec{b}$ there is unique $\vec{x} = A^{-1}\vec{b}$

Pf

- easy to check: $A(A^{-1}\vec{b}) = AA^{-1}\vec{b} = I\cdot\vec{b} = \vec{b}$ ✓
- unique?: Assume $\vec{u}$ solution:

$$A\vec{u} = \vec{b}$$

$$A^{-1}A\vec{u} = A^{-1}\vec{b} \Rightarrow I\vec{u} = A^{-1}\vec{b} \Rightarrow \vec{u} = A^{-1}\vec{b}$$