Linear Algebra

CSCI 2820

Lecture 15

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ECES 122

Today

- Linear and affine vector valued functions, revisited
- Linear systems

Linear Functions

- Vector valued functions (of vectors)

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

$$f(\vec{x}) = (f_1(\vec{x}), \dots, f_m(\vec{x}))$$

$\hookrightarrow$ scalar-valued function of $\vec{x}$

- Matrix-vector product function:

A $m \times n$ matrix. $f : \mathbb{R}^n \to \mathbb{R}^m$ $\boxed{f(\vec{x}) = A\vec{x}}$ (1)

($f(\vec{x}) = \langle a, x \rangle$) special case $m = 1$

- Superposition and linearity: $\boxed{f(\alpha\vec{x} + \beta\vec{y}) = \alpha f(\vec{x}) + \beta f(\vec{y})}$

$$\begin{aligned} f(\alpha\vec{x} + \beta\vec{y}) = A(\alpha\vec{x} + \beta\vec{y}) &= A(\alpha\vec{x}) + A(\beta\vec{y}) = \\ &= \alpha A\vec{x} + \beta A\vec{y} \\ &= \alpha f(\vec{x}) + \beta f(\vec{y}) \end{aligned}$$

Linear Functions

This is the <u>only</u> linear function!

Suppose $f: \mathbb{R}^n \to \mathbb{R}^m$ linear.

we will show there is a matrix A

such that $f(\vec{x}) = A\vec{x}$ for all $\vec{x}$.

<u>Pf</u>. $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$

standard basis of $\mathbb{R}^n$ $\{\vec{e}_1, \dots, \vec{e}_n\}$

$$f(\vec{x}) \underset{\substack{\uparrow \\ \text{linearity}}}{=} \underbrace{\boxed{x_1 f(\vec{e}_1) + x_2 f(\vec{e}_2) + \cdots + x_n f(\vec{e}_n)}}_{\text{can be written as matrix-vector product}} = A\vec{x}$$

$$A = [\underset{\substack{\uparrow \\ m\text{-vectors}}}{f(\vec{e}_1)} \; \dots \; f(\vec{e}_n)]$$

$\boxed{\text{ex: This representation is unique}}$

Linear Functions

Examples. $f: \mathbb{R}^n \to \mathbb{R}^n$

(1) Negation: f changes the sign of $\vec{x}$

$f(\vec{x}) = -\vec{x}$. What is A? $f(\vec{x}) = -I \cdot \vec{x}$

(2) Reversal. f reverses the order of elements of $\vec{x}$. $f(\vec{x}) = (x_n, x_{n-1}, \ldots, x_1)$

what is A? $A = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ & & & \\ 1 & \cdots & & 0 \end{bmatrix}$ (reverser matrix)

(3) Running sum.

$f(\vec{x}) = (x_1, x_1+x_2, \ldots, x_1+x_2+\cdots+x_n)$

$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots 0 & 0 \\ 1 & 1 & 1 & 0 \; 0 & \cdots \\ 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$

Linear Functions

Affine functions $f: \mathbb{R}^n \to \mathbb{R}^m$

$f(\vec{x}) = A\vec{x} + \vec{b}$.

f is affine iff $f(\alpha\vec{x} + \beta\vec{y}) = \alpha f(\vec{x}) + \beta f(\vec{y})$ (restricted superposition)

for all n-vectors $\vec{x}, \vec{y}$, all scalars α, β, $\alpha + \beta = 1$.

$$A = \left[f(\vec{e}_1) - f(\vec{0}) \quad f(\vec{e}_2) - f(\vec{0}) \quad \cdots \quad f(\vec{e}_n) - f(\vec{0}) \right]$$

$\vec{b} = f(\vec{0})$ (exercise)

Linear Functions

(4) De-meaning. $f(\vec{x}) = \vec{x} - avg(\vec{x}) \cdot \vec{1}$

what is A?

$A = I - \frac{1}{n} \cdot J = \begin{bmatrix} 1-1/n & -1/n \cdots & -1/n \\ -1/n & 1-1/n \cdots & \\ & \cdots & \end{bmatrix}$

J = all 1's matrix

Not all functions are linear!

- eg absolute value. $f(\vec{x}) = (|x_1|, |x_2|, \ldots, |x_n|)$

  e.g $n=1$, $x=1$, $y=0$, $\alpha=-1$, $\beta=0$

  $f(\alpha x + \beta y) = f(-1) = 1$

  $\left.\begin{array}{l} \alpha f(x) = -1 \cdot f(1) = -1 \\ \beta f(y) = 0 \end{array}\right\}$ $\alpha f(x) + \beta f(y) = -1 \neq 1$

- "Sort": f sorts the elements of $\vec{x}$ in decreasing order.

  $n=2$, $\vec{x} = (1,0)$, $\vec{y} = (0,1)$, $\alpha = \beta = 1$.

  $f(\alpha\vec{x} + \beta\vec{y}) = f((1,1)) = (1,1)$ while $\begin{array}{l} \alpha f(\vec{x}) = f(1,0) = 1,0 \\ + \beta f(\vec{y}) = f(0,1) = (1,0) \end{array}$ $= (2,0) \neq (1,1)$

Systems of Linear Equations

n-variables $x_1, \dots, x_n$
m linear equations

$$
\left.\begin{aligned}
A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n &= b_1 \\
A_{21}x_1 + A_{22}x_2 + \dots + A_{2n}x_n &= b_2 \\
&\vdots \\
A_{m1}x_1 + A_{m2}x_2 + \dots + A_{mn}x_n &= b_m
\end{aligned}\right\} \quad A\vec{x} = \vec{b}
$$

coefficient matrix

right hand side

A_{ij} coefficients
b_i "right hand sides"

$\vec{x}$ is solution if
$A\vec{x} = \vec{b}$

eg:

$$
\begin{aligned}
x_1 + x_2 &= 1 \\
x_1 &= -1 \\
x_1 - x_2 &= 0
\end{aligned}
, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}
$$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ potential sol

$$
\left.\begin{aligned}
x_1 = x_2 &= -1 \\
x_1 + x_2 &= 1
\end{aligned}\right] \text{impossible}
$$

no sol

Systems of Linear Equations

- eg2 :
$$\left.\begin{array}{l} x_1+x_2=1 \\ x_2+x_3=2 \end{array}\right\} A\vec{x}=\vec{b}$$

$$A=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \vec{b}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Infinite # sol. eg $(1,0,2)$ $\left(\begin{array}{l} x_1+x_2=1 \\ x_2+x_3=2 \end{array}\right)$

$(0,1,1)$ $\left(\begin{array}{l} x_1+x_2=1 \\ x_2+x_3=2 \end{array}\right)$

- system of linear equations is

(a) over-determined if $m>n$ $\iff$ coeff. matrix tall

(b) under-determined if $m<n$ — wide

(c) square is $m=n$ — square

(d) $A\vec{x}=\vec{0}$ homogeneous, always $\vec{x}=\vec{0}$ is solution

Systems of Linear Equations

(a) Polynomial Interpolation:

poly p, degree $\le n-1$ that interpolates m given points (t_i, y_i) $i = 1, \dots, m)$
$(p(t_i) = y_i)$.

$$p(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$$

$$A\vec{c} = \vec{y}, \quad A = \begin{bmatrix} 1 & t_1 & \dots & t_1^{n-2} & t_1^{n-1} \\ 1 & t_2 & \dots & t_2^{n-2} & t_n^{n-1} \\ \vdots & & & & \\ 1 & t_m & \dots & t_m^{n-2} & t_m^{n-1} \end{bmatrix}$$

$$\vec{c} = \begin{pmatrix} c_0 \\ \vdots \\ c_{n-1} \end{pmatrix}$$

Systems of Linear Equations

Balancing chemical reactions

$a_1 R_1 + \cdots + a_p R_p \longrightarrow b_1 P_1 + \cdots + b_q P_q$

e.g. $\boxed{2H_2O \rightarrow 2H_2 + O_2}$

water electrolysis

balancing atoms $\rightarrow$ m equations (m is # of different atoms)

$m \times p$ matrix R. R_{ij} = # atoms of type i in R_j

$\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix}$ $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_q \end{pmatrix}$. $(R\vec{a})_i$ = total number of atoms of type i appearing in the reactants.

$m \times q$ matrix P (similarly)

$(P\vec{b})_i$ = total number of atoms of type i in products.

$R\vec{a} = P\vec{b} \Rightarrow \boxed{\begin{bmatrix} R & -P \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} = \vec{0}}$

Examples

enforce a non-zero sol. by taking

$a_1 = 1$;

$$\begin{bmatrix} R & -P \\ e_1^T & 0 \end{bmatrix} \begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix} = \vec{e_{m+1}}$$

$m+1$ eq, $p+q$ var.

eg electrolysis, $p = 1$ reactant
$q = 2$ products
$m = 2$ atoms

$$R = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -2 & 0 \\ 1 & 0 & -2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ b_2 \end{bmatrix}$$

$(1, 1, 1/2)$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Examples

Examples

Practice Problems

Cross-product. The cross product of two 3-vectors $a = (a_1, a_2, a_3)$ and $x = (x_1, x_2, x_3)$ is defined as the vector

$$a \times x = \begin{bmatrix} a_2x_3 - a_3x_2 \\ a_3x_1 - a_1x_3 \\ a_1x_2 - a_2x_1 \end{bmatrix}.$$

The cross product comes up in physics, for example in electricity and magnetism, and in dynamics of mechanical systems like robots or satellites. (You do not need to know this for this exercise.)

Assume a is fixed. Show that the function $f(x) = a \times x$ is a linear function of x, by giving a matrix A that satisfies $f(x) = Ax$ for all x.

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_2x_3 - a_3x_2 \\ a_3x_1 - a_1x_3 \\ a_1x_2 - a_2x_1 \end{bmatrix}$$

Practice Problems

Practice Problems