Linear Algebra

CSCI 2820

Lecture 16

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ECES 122

Today

- Matrix Multiplication, Revisited
- Composition of vector valued linear functions
- QR factorization

Matrix-Matrix multiplication reminder

$A_{m\times n}$, $B_{n\times k}$

$C = A \cdot B$, $\dim \stackrel{?}{=} m\times k$

$$\boxed{C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}}$$ (*)

- scalar-vector product $\underset{n\times 1}{\vec{x}} \cdot \underset{1\times 1}{a} = \vec{y}_{n\times 1}$ $(a\vec{x})$
- inner product $\langle x, y \rangle$ $1\times m$, $m\times 1$ = scalar
- Matrix-vector $\vec{y} = A\vec{x}$, $A_{m\times n}$, $\vec{x}$ $n\times 1$ $\vec{y}$ = $m\times 1$ dim
- outer product m-vector $\vec{a}$ n vector $\vec{b}$: $\underset{m\times 1}{\vec{a}}\,\underset{1\times n}{\vec{b}^T}$: $m\times n$ matrix

$$\begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ \vdots & & & \\ a_mb_1 & \cdots & & a_mb_n \end{bmatrix}$$

$\vec{a}\vec{b}^T \neq \vec{b}\vec{a}^T$

Matrix-Matrix multiplication reminder

- In general, $AB \neq BA$
- $(AB)^T = B^T A^T$

<u>Inner product</u>: $\vec{y}, \vec{x}, A$

$\langle y, Ax \rangle = \langle A^T y, x \rangle$

Pf: $\vec{y}^T (A\vec{x}) = (\vec{y}^T A) \cdot \vec{x} = (A^T y)^T \cdot \vec{x}$

$(c \cdot A) \cdot B$

$= (\langle A^T y, x \rangle)$

- Column interpretation:

$A, \quad B = [\vec{b}_1 \; \ldots \; \vec{b}_n]$

$AB = [A\vec{b}_1 \quad A\vec{b}_n]$

- Linear equations — multiple sets of linear equations

x: $A\vec{x}_i = \vec{b}_i, \; i = 1 \ldots k$

$\boxed{AX = B}$

$X = [\vec{x}_1 \ldots \vec{x}_k], \; B = [\vec{b}_1 \ldots \vec{b}_k]$

Matrix-Matrix multiplication reminder

- Inner product representation

$$AB = \begin{bmatrix} \vec{a}_1^T \vec{b}_1 & \vec{a}_1^T \vec{b}_2 & \cdots & \vec{a}_1^T \vec{b}_n \\ \vdots & & & \\ \vec{a}_m^T \vec{b}_1 & \vec{a}_m^T \vec{b}_2 & \cdots & \vec{a}_m^T \vec{b}_n \end{bmatrix}$$

- Gram matrix $\quad A_{m\times n} \quad A = [\vec{a}_1 \cdots \vec{a}_n]$

$$G = A^T A = \begin{bmatrix} \vec{a}_1^T \vec{a}_1 & \vec{a}_1^T \vec{a}_2 & \cdots & \vec{a}_1^T \vec{a}_n \\ \vdots & & & \\ \vec{a}_n^T \vec{a}_1 & \vec{a}_n^T \vec{a}_2 & \cdots & \vec{a}_n^T \vec{a}_n \end{bmatrix}$$

- symmetric
$\vec{a}_i^T a_j = \vec{a}_j^T \vec{a}_i$

e.g. $A_{m\times n}$: $\quad A_{ij} = \begin{cases} 1 & \text{if item } i \text{ in group } j \\ 0 & \text{o.w.} \end{cases}$

$G = A^T A$: $\quad G_{ij}$ = number of items both in group i & j

G_{ii} = number of items in group i.

Outer Product, Gram Matrix

- Outer product interpretation

$A = [\vec{a}_1 \ \dots \ \vec{a}_p]$, $B = \begin{bmatrix} \vec{b}_1^T \\ \vdots \\ \vec{b}_p^T \end{bmatrix}$

$A \cdot B = \vec{a}_1 \vec{b}_1^T + \dots + \vec{a}_p \vec{b}_p^T$

Outer Product, Gram Matrix

Outer Product, Gram Matrix

Bra/Ket notation

$\langle x, y \rangle$: inner product (scalar)

$|x\rangle\langle y|$: outer product (matrix)

$|x\rangle \overset{\text{(ket)}}{=}$ column vector

$\langle y| \overset{\text{bra}}{=}$ row vector $(\vec{y}^T)$

$\vec{b} \cdot \vec{a}^T \cdot \vec{c} \cdot \vec{d}^T \overset{?}{=}$

$|b\rangle \underbrace{\langle a|c\rangle}_{\text{inner product (scalar)}} \langle d| = \langle a|c\rangle \cdot \underbrace{|b\rangle\langle d|}_{\text{outer product (matrix)}}$

Matrix products as Composition of Linear Functions

$f(\vec{x}) = A\vec{x}$ $\quad f : \mathbb{R}^p \to \mathbb{R}^m$

$g(\vec{x}) = B\vec{x}$ $\quad g : \mathbb{R}^n \to \mathbb{R}^p$

composition of f, g: $h : \mathbb{R}^n \to \mathbb{R}^m$

$$h(\vec{x}) = f(g(\vec{x})) = A(B\vec{x}) = \underbrace{(AB)}_{C}\vec{x}$$

$$\underbrace{f(\underbrace{p\text{-dim}})}_{m\text{-dim}}$$

e.g of composition of linear functions: $(n-1)\times n$ difference matrix D_n

$$D_n = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & & \cdots & -1 & 1 \end{bmatrix}, \quad D_n\vec{x} = (x_2 - x_1, \ldots, x_n - x_{n-1})$$

D_{n-1} : $(n-2)\times(n-1)$ matrix

$D_{n-1}D_n$ = second difference matrix

Matrix products as Composition of Linear Functions

- *Difference matrix.* The $(n-1) \times n$ matrix

$$D_n = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

$$D_{n-1} D_n \vec{x} = D_{n-1} \left(x_2 - x_1, \ldots, x_n - x_{n-1} \right) = \left(x_1 - 2x_2 + x_3, \cdots, x_{n-2} - 2x_{n-1} + x_n \right)$$

Matrix products as Composition of Linear Functions

Composition of Affine functions

$f : \mathbb{R}^p \to \mathbb{R}^m \qquad f(\vec{x}) = A\vec{x} + \vec{b}$

$g : \mathbb{R}^n \to \mathbb{R}^p \qquad g(\vec{x}) = C\vec{x} + \vec{d}$

$$h(\vec{x}) = f(g(\vec{x})) = A(C\vec{x} + \vec{d}) + \vec{b} = \underbrace{(AC)}_{\tilde{A}}\vec{x} + \underbrace{(A\vec{d} + \vec{b})}_{\tilde{b}}$$

$$h(\vec{x}) = \tilde{A}\vec{x} + \tilde{b}$$

Matrix Powers and graphs

$$\underbrace{A \cdot A \cdots \cdot A}_{k} = A^k \qquad , \quad A^{1/2} \quad A^0 = I$$

$(A^{-1})^k$: inverse

$$A^k \cdot A^\ell = A^{k+\ell}$$

$$(A^k)^\ell = A^{k \cdot \ell}$$

$A_{n \times n}$: adjacency matrix

$$A_{ij} = \begin{cases} 1 & \text{if } j \to i \\ 0 & \text{ow} \end{cases}$$

. path of length ℓ : sequence of $\ell+1$ vertices

$$(1) \to (2) \to (3) \to \cdots \to (\ell+1)$$

$$j \to k \to i$$

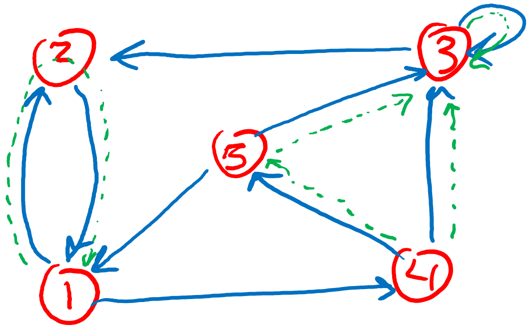
$$(A^2)_{ij} = \sum_{k=1}^{n} \underbrace{\overset{=1}{A_{ik}} \overset{=1}{A_{kj}}}$$

$\neq 0$ iff $\exists$ path of length 2 between j and i

= # paths of length 2 from $j \rightsquigarrow i$

Matrix Powers and graphs

Matrix Powers and graphs



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$(A^2)_{11} = 1$ (1 path (1, 2, 1))

$(A^2)_{34} = 2$ (2 paths (4,5,3) and (4,3,3))

$(A^\ell)_{ij}$ = # paths of length ℓ from vertex j to vertex i

Pf by induction: . Base case : A^2 ✓

IH: Assume true that $(A^k)_{ij}$ = # paths $j \leadsto i$ $\forall$ $k \le \ell$

need to show it holds for $\ell+1$. $(A^{\ell+1})_{ij} = (A \cdot A^\ell)_{ij} = \sum_{k=1}^{m} \underbrace{A_{ik}(A^\ell)_{kj}}$

I.H. # length ℓ paths $j \to k$

k-th term = # length ℓ paths $j \leadsto k$ iff edge $k \to i$
= # length $\ell+1$ paths $j \leadsto i$ that end with edge $k \to i$ } sum over all k ✓

QR factorization

Algorithm 5.1 GRAM–SCHMIDT ALGORITHM

given n-vectors $a_1, \ldots, a_k$

for $i = 1, \ldots, k$,

1. *Orthogonalization.* $\tilde{q}_i = a_i - (q_1^T a_i)q_1 - \cdots - (q_{i-1}^T a_i)q_{i-1}$
2. *Test for linear dependence.* if $\tilde{q}_i = 0$, quit.
3. *Normalization.* $q_i = \tilde{q}_i / \|\tilde{q}_i\|$

- Matrices with orthonormal columns:

$\{\vec{a}_1, \ldots, \vec{a}_k\}$ orthonormal : $\boxed{A^T A = I}$ if A is square then A is called orthogonal

$A = [\vec{a}_1 \ \ldots \ \vec{a}_k]$

ex: $A_{n \times n}$ with orthonormal columns also has orthonormal rows.

QR factorization

<u>Norm, inner product, angle.</u>

$A_{m\times n}$, orthonormal colums, $\vec{x}, \vec{y}$ n-vectors

(1) $\|A\vec{x}\| = \|\vec{x}\|$

(2) $\langle Ax, Ay\rangle = \langle x, y\rangle$

(3) $\angle(A\vec{x}, A\vec{y}) = \angle(\vec{x}, \vec{y})$

$(AB)^T = B^T A^T$

Pf: (2) $(A\vec{x})^T (A\vec{y}) = (\vec{x}^T A^T)(A\vec{y}) = \vec{x}^T \underbrace{(A^T A)}_{\text{inner product property}} \vec{y} = \vec{x}^T I \vec{y} = \vec{x}^T \vec{y}$

(2) $\rightarrow$ (1) $\vec{y} = \vec{x}$

$(A\vec{x})^T (A\vec{x}) = \vec{x}^T \vec{x}$ by (2)

$\|A\vec{x}\|^2 = \|\vec{x}\|^2 \Rightarrow \|A\vec{x}\| = \|\vec{x}\|$

(1),(2) $\rightarrow$ (3): $\angle(A\vec{x}, A\vec{y}) = \arccos\left(\frac{(A\vec{x})^T (A\vec{y})}{\|A\vec{x}\|\|A\vec{y}\|}\right) = \arccos\frac{\vec{x}^T y}{\|\vec{x}\|}$

Practice Problems

Matrix sizes. Suppose A, B, and C are matrices that satisfy $A + BB^T = C$. Determine which of the following statements are necessarily true. (There may be more than one true statement.)

(a) A is square.

(b) A and B have the same dimensions.

(c) A, B, and C have the same number of rows.

(d) B is a tall matrix.

Practice Problems

When is the outer product symmetric? Let a and b be n-vectors. The inner product is symmetric, *i.e.*, we have $a^Tb = b^Ta$. The outer product of the two vectors is generally *not* symmetric; that is, we generally have $ab^T \neq ba^T$. What are the conditions on a and b under which $ab = ba^T$? You can assume that all the entries of a and b are nonzero. (The conclusion you come to will hold even when some entries of a or b are zero.) *Hint.* Show that $ab^T = ba^T$ implies that a_i/b_i is a constant (*i.e.*, independent of i).