Linear Algebra

CSCI 2820

Lecture 17

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Practice Problems

Matrix sizes. Suppose A, B, and C are matrices that satisfy $A + \underbrace{BB^T}_{n \times n} = C$. Determine which of the following statements are necessarily true. (There may be more than one true statement.)

(a) A is square. ✓

(b) A and B have the same dimensions.

(c) A, B, and C have the same number of rows. ✓

(d) B is a tall matrix.

$A : n \times n$

$B : n \times m$, $B^T : m \times n$, $BB^T : n \times n$

$C : n \times n$

Practice Problems

When is the outer product symmetric? Let a and b be n-vectors. The inner product is symmetric, *i.e.*, we have $a^Tb = b^Ta$. The outer product of the two vectors is generally *not* symmetric; that is, we generally have $ab^T \neq ba^T$. What are the conditions on a and b under which $ab = ba^T$? You can assume that all the entries of a and b are nonzero. (The conclusion you come to will hold even when some entries of a or b are zero.) *Hint.* Show that $ab^T = ba^T$ implies that a_i/b_i is a constant (*i.e.*, independent of i).

$\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_{n\times 1}$, $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}_{n\times 1}$ $\vec{a}^T, \vec{b}^T$ $_{1\times n}$

$\vec{a}\,\vec{b}^T = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}(b_1, \ldots b_n)$

$= \begin{bmatrix} a_1b_1 & a_1b_2 & \ldots & a_1b_n \\ \vdots & & & \\ a_nb_1 & \ldots & \ldots & a_nb_n \end{bmatrix}$

$\vec{b}\,\vec{a}^T = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}(a_1 \ldots a_n)$

$= \begin{bmatrix} b_1a_1 & b_1a_2 & \ldots & b_1a_n \\ \vdots & & \ddots & \vdots \\ b_na_1 & \ldots & \ldots & b_na_n \end{bmatrix}$

$|a\rangle\langle b|e_j\rangle = |b\rangle\langle a|e_j\rangle$

$\boxed{b_j\,|a\rangle = a_j\,|b\rangle} \;\forall j$

$\boxed{|a\rangle = \frac{a_j}{b_j}|b\rangle, \quad \frac{a_j}{b_j} = c}$

$a_1b_2 = b_1a_2 \quad \ldots \quad a_ib_j = b_ia_j \Rightarrow \frac{a_i}{b_i} = \frac{a_j}{b_j} = c \quad \forall i,j$

$\boxed{\forall i:\; a_i = c\cdot b_i \Rightarrow \vec{a} = c\,\vec{b}}$

Today

- QR factorization
- Left Inverse
- Right Inverse
- Inverse

QR factorization

Algorithm 5.1 GRAM–SCHMIDT ALGORITHM

given n-vectors $a_1, \ldots, a_k$

for $i = 1, \ldots, k$,

1. *Orthogonalization.* $\tilde{q}_i = a_i - (q_1^T a_i)q_1 - \cdots - (q_{i-1}^T a_i)q_{i-1}$
2. *Test for linear dependence.* if $\tilde{q}_i = 0$, quit.
3. *Normalization.* $\boxed{q_i = \tilde{q}_i / \|\tilde{q}_i\|}$

$A = [\vec{a}_1 \; \cdots \; \vec{a}_k]$ assume $\{\vec{a}_i\}$ linearly indep.

$Q = [\vec{q}_1 \; \cdots \; \vec{q}_k] \Rightarrow \boxed{Q^T Q = I}$

(1): $\vec{a}_i = \underbrace{\langle q_1, a_i \rangle}_{R_{1i}} \vec{q}_1 + \cdots + \underbrace{\langle q_{i-1}, a_i \rangle}_{R_{i-1\,i}} \vec{q}_{i-1} + \underbrace{\|\tilde{q}_i\|}_{R_{ii}} \vec{q}_i$

$R = [R_{ij}]_{ij}$

$R_{ij} = \langle q_i, a_j \rangle$, $i<j$ and $R_{ii} = \|\tilde{q}_i\|$, $\underline{R_{ij} = 0}$ $i>j$

R is upper-triangular by def:

$A = QR$ → upper-triangular (R)
↳ orthonormal columns (Q)

QR factorization

Left Inverse of matrix

$\underset{m\times n}{A}$, $\underset{n\times m}{X}$ is left-inverse of A if $X \cdot A = I_{n\times n}$

eg. · a, $a^{-1} = 1/a$ $\quad a \cdot 1/a = 1 \quad \forall a \in \mathbb{R} - \{0\}$

· n-vector $\vec{a}$ (nx1 matrix) is left-invertible

$\vec{x}_i = \frac{1}{a_i} e_i^T$ (1xn matrix) $\quad \therefore \vec{x}_i \cdot \vec{a} \overset{?}{=} 1$ ✓

$a_i \neq 0$ $\quad \frac{1}{a_i} \cdot a_i = 1$

· $A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$, $B = \frac{1}{9}\begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix}$,

$C = \frac{1}{2}\begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix}$

check: $BA = CA = I$

Left Inverse of matrix

ex: If A has more than one left inverse, then it has infinitely many.

$$BA = CA = I$$

$$(B \cdot C)A = B \cdot (CA) = B \cdot I = B \quad \times$$

$$\left(\tfrac{1}{2}B + \tfrac{1}{2}C\right)A = \tfrac{1}{2}BA + \tfrac{1}{2}CA = \tfrac{1}{2}I + \tfrac{1}{2}I = I \quad \checkmark$$

$\boxed{\alpha B + \beta C}$: any affine combination.

$\alpha + \beta = 1$

$$(\alpha B + \beta C)A = \alpha BA + \beta CA = \alpha I + \beta I = (\alpha + \beta)I = I$$

- $A^T A = I$, A^T left inverse.

Left Inverse of matrix

Claim: If A has a left inverse C, then the columns of A are linearly indep.

Pf: Suppose $\underbrace{\boxed{x_1 \cdot \vec{a}_1 + \dots + x_n \vec{a}_n = \vec{0}}}$ iff all $x_i = 0$

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$A \cdot \vec{x} = \vec{0}$$

$$C \cdot (A \cdot \vec{x}) = \vec{0} \Rightarrow$$

$$\underbrace{(C \cdot A)}_{I} \vec{x} = \vec{0} \Rightarrow I\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$$

"A matrix has left inverse iff its columns are L.I"

Note: $A_{m \times n}$, $m < n$: does it have left inverse? NO!

- Solving linear systems: $A\vec{x} = \vec{b}$, A $m \times n$ tall or square, $CA = I$

"over-determined" $C(A\vec{x}) = C\vec{b} \Rightarrow (CA)\vec{x} = C\vec{b} \Rightarrow \boxed{\vec{x} = C\vec{b}}$

Right Inverse of matrix

- X is right inverse of A if: $AX = I$
- If A has right inverse B, then B^T is a left inverse of A^T $B^TA^T = (AB)^T = I$

and if A has left inverse C, then C^T is right inverse of A^T, $A^TC^T = (CA)^T = I$.

- A matrix is right-invertible iff its rows are L.I.
- A tall matrix cannot have a right inverse

Right Inverse of matrix

Solving linear equations:

m linear equations, n vars.

$\boxed{A\vec{x} = \vec{b}}$, B is a right inverse of A

(under-determined or square)

then for **any** $\vec{b}$: $\boxed{\vec{x} = B\vec{b}}$.

$$A\vec{x} = AB\vec{b} = I\vec{b} = \vec{b}$$

Examples:

$$B = \frac{1}{9}\begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$$

$$C = \frac{1}{2}\begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix}$$

$\boxed{A\vec{x} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}}$, $\vec{x} = B\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ check $A \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ ✓

or $C\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Right Inverse of matrix

$A\vec{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, check $C\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$

not a solution

$A\vec{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ does not have sol.

- Under-determined system: $\underset{2\times 3}{A^T}\vec{y} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$B^T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = (1/3, 2/3, 38/9)$

$C^T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = (0, 1/2, -1)$

Product: A, D. A has right-inverse B, D has right inverse E

then $(AD)\cdot(EB) = A(DE)B = A\cdot I\cdot B = A\cdot B = I$

A left inverse C, D has left inverse F. $(FC)\cdot(AD) = I$.

Inverse of matrix

A right + left invertible then those inverses are <u>unique</u> + <u>equal</u> (called inverse, A^{-1})

$AX = I$: $X = \underbrace{(YA)}_{I}.X = Y\underbrace{(AX)}_{I} = Y$

$YA = I$

$\boxed{A \text{ is invertible or non-singular + square}}$

$AA^{-1} = A^{-1}A = I$, $(A^{-1})^{-1} = A$, A, A^{-1} are inverses of each other.

- <u>solving linear equations (square)</u>

$A\vec{x} = \vec{b}$, A invertible : $\boxed{\vec{x} = A^{-1}\vec{b}}$ for any $\vec{b}$ (A^{-1} right inverse) ✓

+ only solution (since A^{-1} is left inverse) ✓

$\Rightarrow$ solution of square + invertible system is linear function of $\vec{b}$

Inverse of matrix

Square matrices: left-invert, right invert, invert are equivalent: If matrix square + left-invertible ⇒ right-invertible ⇒ invertible.

Pf: $A_{n \times n}$, left-inv. ⇒ colums L.I. (basis)

$\forall$ vector in $\mathbb{R}^n$ can be written as linear comb of $\vec{a_i}$

$A = [\vec{a_1} \; \cdots \; \vec{a_n}]$

In particular, $\vec{e_i} = A\vec{b_i}$ $\quad [\vec{e_i} = b_i(1)\vec{a_1} + \cdots + b_i(n)\vec{a_n}]$

$B = [\vec{b_1} \; \cdots \; \vec{b_n}]$:

$$AB = [A\vec{b_1} \; \cdots \; A\vec{b_n}] = [\vec{e_1} \cdots \vec{e_n}] = I$$

⇒ B is right-inverse of A.

left invert ⇒ colum indep ⇒ right invert (+ row indep)

A^T: right invert ⇒ row indp ⇒ left invert (+ colum indep)

Inverse of matrix

Inverse of matrix