Linear Algebra

CSCI 2820

Lecture 18

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ECES 122

Today

- More on Inverses
- Solving Systems of Equations
- Computing the inverse
- Pseudoinverse

Inverse examples

Reminder: T.F.A.E: (for n×n matrix)

- A is invertible
- A has l. ind. columns
- A has l. ind. rows
- A has left inverse
- A has right inverse.

eg. 1: I, $I^{-1} = I$

$I \cdot I = I$

eg2: $A = \begin{bmatrix} a_1 & 0 & 0 & \cdots \\ 0 & \ddots & & \\ 0 & & & \\ 0 & & & a_n \end{bmatrix}$ ←

diag$(a_1, \dots, a_n)$

when is it invertible? all a_i's $\neq 0$

$A^{-1} = \begin{bmatrix} 1/a_1 & & 0 \\ 0 & \ddots & 1/a_n \end{bmatrix}$

$= \text{diag}(1/a_1, \dots, 1/a_n)$

eg3. $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 2 \\ -3 & -4 & -4 \end{bmatrix}$, $A^{-1} = \frac{1}{30} \begin{bmatrix} 0 & -20 & -10 \\ -6 & 5 & -2 \\ 6 & 10 & 2 \end{bmatrix}$

check $AA^{-1} = A^{-1}A = I$.

eg4: 2x2 matrices : $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$

($A_{11}A_{22} - A_{12}A_{21}$: determinant)

invertible iff: $A_{11}A_{22} - A_{12}A_{21} \neq 0$

eg5: orthogonal matrix: $A^T A = I$, $A^{-1} = A^T$

Inverse examples

- **Inverse of Transpose:**

$$(A^T)^{-1} = (A^{-1})^T \left(= A^{-T} \right)$$

- **Inverse of product:** A, B $n \times n$ invertible

$$\boxed{(AB)^{-1} = B^{-1} \cdot A^{-1}}$$ $\left(\text{check: } AB \cdot B^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I \right)$

Question: Can A, B ($n \times n$) s.t. either A or B not invertible, yet AB invertible? (ex. for home)

- **Dual basis:** A $n \times n$ invertible, $B = A^{-1}$, $A = [\vec{a}_1 \dots \vec{a}_n]$

$\Rightarrow \underbrace{\{\vec{a}_1, \dots, \vec{a}_n\}}_{B_1}$ basis

$$B = \begin{bmatrix} b_1^T \\ \vdots \\ b_n^T \end{bmatrix} \Rightarrow \underbrace{\{\vec{b}_1, \dots, \vec{b}_n\}}_{B_2} \text{ basis}$$

B_1, B_2 are dual bases

$\vec{x} = \beta_1 \vec{a}_1 + \dots + \beta_n \vec{a}_n$: Find β_i's via dual basis B_2

Inverse examples

$AB = I$

$$\vec{x} = \underbrace{AB}_{I}\vec{x} = [\vec{a}_1 \cdots \vec{a}_n]\begin{bmatrix}\vec{b}_1^T \\ \vdots \\ \vec{b}_n^T\end{bmatrix}\vec{x}$$

$$= \underbrace{\langle b_1, x\rangle}_{\beta_1}\vec{a}_1 + \cdots + \underbrace{\langle b_n, x\rangle}_{\beta_n}\vec{a}_n$$

$$\beta_i = \langle b_i, x\rangle$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = [\vec{a}_1 \; \vec{a}_2]$$

$$B = [\vec{a}_1 \; \vec{a}_2]^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \quad \begin{matrix} = b_1^T \\ = b_2^T \end{matrix}$$

Quick note

A $n \times n$ invertible, $k \in \mathbb{N}$

$$(A^k)^{-1} = (A^{-1})^k = A^{-k}$$

$A^2 = A \cdot A$

$$(A^2)^{-1} = (A \cdot A)^{-1} = A^{-1} \cdot A^{-1} = (A^{-1})^2 = A^{-2}$$

$\{\vec{a}_1, \vec{a}_2\}$ and $\{\vec{b}_1, \vec{b}_2\}$ dual bases

$$\vec{x} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}, \quad \vec{x} = \beta_1\vec{a}_1 + \beta_2\vec{a}_2$$

$$\beta_1 = \langle b_1, x\rangle = -2$$

$$\beta_2 = \langle b_2, x\rangle = -3$$

check: $\begin{pmatrix} -5 \\ 1 \end{pmatrix} = -2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -1 \end{pmatrix} \overset{?}{=} \begin{pmatrix} -5 \\ 1 \end{pmatrix}$

Triangular Matrix

Claim: L triangular matrix is invert. iff it has non-zero diag. elements.

Proof: assume $L_{n\times n}$ lower triangular, we will show L.I columns. I.e. $L\vec{x}=\vec{0} \Leftrightarrow \vec{x}=\vec{0}$

$$L\vec{x} = \vec{0} \Leftrightarrow L_{11}x_1 = 0 \Rightarrow x_1 = 0 \; (L_{11} \neq 0)$$

$$L = \begin{bmatrix} L_{11} & & & \\ L_{21} & L_{22} & & \emptyset \\ \vdots & & \ddots & \\ L_{n1} & \cdots & \cdots & L_{nn} \end{bmatrix}$$

$$L_{21}x_1 + L_{22}x_2 = 0 \Rightarrow x_2 = 0$$

$$L_{31}x_1 + L_{32}x_2 + L_{33}x_3 = 0 \Rightarrow x_3 = 0 \quad \text{etc}$$

$$\vdots$$

$$\vec{x} = \vec{0}$$

Upper triangular R: repeat proof or $L = R^T$ lower triang.

R^T invertible by previous proof, $(R^T)^T$ invertible.

Inverse via QR factorization

$A = Q \cdot R$ (assume A $n \times n$ invertible)

Q ↓ orthogonal

R ↳ upper-triangular

$A^{-1} = (QR)^{-1} = R^{-1} Q^{-1} = R^{-1} \cdot Q^{T}$

R^{-1} → algorithm shortly

Q^{T} → easy

Solving linear equations

$R\vec{x} = \vec{b}$, R upper-triangular $n \times n$
non-zero diag.

Back substitution:

$R_{11} x_1 + R_{12} x_2 + \dots + R_{1n} x_n = b_1$

$\vdots$

$R_{n-1,n-1} x_{n-1} + R_{n-1,n} x_n = b_{n-1}$ $\quad x_{n-1} = (b_{n-1} - R_{n-1,n} x_n) / R_{n-1,n-1}$

$R_{nn} x_n = b_n \Rightarrow x_n = b_n / R_{nn}$

algo: for $i = n, \dots, 1$: $x_i = (b_i - R_{i,i+1} x_{i+1} \dots - R_{i,n} x_n) / R_{ii}$

computes the solution of $R\vec{x} = \vec{b}$, $\vec{x} = R^{-1} \vec{b}$

Solving linear equations-QR

Algorithm 11.2 SOLVING LINEAR EQUATIONS VIA QR FACTORIZATION

given an $n \times n$ invertible matrix A and an n-vector b.

1. *QR factorization.* Compute the QR factorization $A = QR$.
2. Compute $Q^T b$. $= \vec{y}$
3. *Back substitution.* Solve the triangular equation $Rx = Q^T b$ ($\vec{y}$) using back substitution.

$(A)^{-1} = (QR)^{-1} = R^{-1} Q^T$. Method for solving linear eq. for any $n \times n$ invert A. $A\vec{x} = \vec{b}$.

solution is $\vec{x} = A^{-1}\vec{b} = R^{-1} Q^T \vec{b}$

compute $Q^T \vec{b} = \vec{y}$

solve upper-triang. system $R\vec{x} = \vec{y}$ by back-sub.

Solving linear equations

Solving linear equations

$A\vec{x}_1 = \vec{b}_1$

$A\vec{x}_2 = \vec{b}_2$

$\vdots$

$A\vec{x}_k = \vec{b}_k$

$AX = B$, A invertible

$X = [\vec{x}_1 \dots \vec{x}_k]$

$B = [\vec{b}_1 \dots \vec{b}_k]$

$X = A^{-1}B$ solution

faster method: $A = QR$ (11.3)

$\vec{y}_k = Q^T\vec{b}_k$, $R\vec{x}_k = \vec{y}_k$

Computing matrix inverse

Algorithm 11.3 COMPUTING THE INVERSE VIA QR FACTORIZATION

given an $n \times n$ invertible matrix A.

1. *QR factorization.* Compute the QR factorization $A = QR$.
2. For $i = 1, \ldots, n$,
   Solve the triangular equation $Rb_i = \tilde{q}_i$ using back substitution.

A n×n invertible, $\underbrace{A^{-1}}_{B} = ?$ $A = QR$, $A^{-1} = R^{-1} Q^T \Rightarrow$

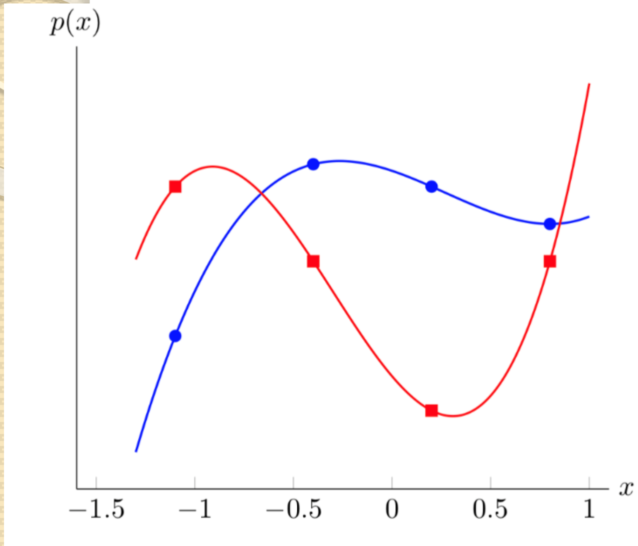
$$\boxed{R \cdot B = Q^T}$$

$\{ R \vec{b}_i = \vec{q}_i \}$ — $\vec{b}_i$ columns of B, $\vec{q}_i$ columns of Q^T

do at home: read section 11.4 (how to compute inverses)

Computing matrix inverse

Examples



Examples

Pseudo-inverse

- $m \times n$ matrix A l. indep columns (tall/square)
  iff its $n \times n$ Gram matrix $A^T A$ is invertible

$\Rightarrow$ columns of A l.i. let $\vec{x}$ n-vector

$(A^T A)\vec{x} = \vec{0} \Rightarrow$. Need to show $\vec{x} = \vec{0}$

$\vec{x}^T (A^T A)\vec{x} = 0 \Rightarrow$

$\vec{x}^T A^T A \vec{x} = 0 \Rightarrow \|A\vec{x}\|^2 = 0 \Rightarrow A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$ (implies $A^T A$ invert)

$\Leftarrow$ columns of A are l. dependent. Need to show

$A^T A$ is singular.

$\exists\, \vec{x} \neq \vec{0}$ st $A\vec{x} = \vec{0} \Rightarrow (A^T A)\vec{x} = \vec{0} \Rightarrow$ columns of $A^T A$ L.D, $A^T A$ singular

Pseudo-inverse

Claim: If A (square or tall) has linearly indep. columns then it has a left inverse.

Proof: Assume A has L.I columns, then A^TA is invertible. Observe $(A^TA)^{-1}.A^T$ is a left-inverse of A

check: $\left[(A^TA)^{-1}\cdot A^T\right]A = (A^TA)^{-1}(A^TA) = I$

pseudo-inverse (Moore-Penrose inverse) A^{+}

if A square $A^{+} = A^{-1}$: $A^{+} = (A^TA)^{-1}A^T = A^{-1}A^{-T}A^T = A^{-1}$

Pseudo-inverse

Transpose all equations for wide matrices.

if a wide matrix A has l.I rows then it has right invers

$A^T(AA^T)^{-1}$, pseudo inverse, A^+

$A\vec{x}=\vec{b}$ systems

- over-determined systems with A L.I columns, and $\exists$ solution: $\vec{x}=A^+\vec{b}$
- under-det: rows of A L.I, $\vec{x}=A^+\vec{b}$.

Pseudo-inverse

Exercise

11.5 *Inverse of a block matrix.* Consider the $(n+1) \times (n+1)$ matrix

$$A = \begin{bmatrix} I & a \\ a^T & 0 \end{bmatrix},$$

where a is an n-vector.

(a) When is A invertible? Give your answer in terms of a. Justify your answer.

(b) Assuming the condition you found in part (a) holds, give an expression for the inverse matrix A^{-1}.