# Linear Algebra

## CSCI 2820

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ECES 122

# Today

- Linear combinations
- Inner product
- Complexity of vector operations
- Examples

# Linear combinations of Vectors

- <u>sum</u> : $\beta_1 = \cdots = \beta_m = 1$ , $\vec{a}_1 + \cdots + \vec{a}_m$
- <u>average</u> : $\beta_1 = \cdots = \beta_m = 1/m$ , $\frac{1}{m}(\vec{a}_1 + \cdots + \vec{a}_m)$
- <u>affine</u> : $\beta_1 + \beta_2 + \cdots + \beta_m = 1$

  if all +ve $\rightarrow$ weighted average

<u>ex</u>

$\vec{a}_i$ audio signal.

$\beta_1 \vec{a}_1 + \cdots + \beta_m \vec{a}_m$

# Linear combinations of Vectors

vector : $\vec{a}$, $|a\rangle$

$\vec{a}_1, \dots, \vec{a}_m$ (or, $|a_1\rangle \dots |a_m\rangle$) dim $n$

→ $\beta_1, \dots, \beta_m$ scalars

Def: linear combination of $\vec{a}_1, \dots, \vec{a}_m$

is $\beta_1 \vec{a}_1 + \dots + \beta_m \vec{a}_m = \vec{k}$

same dim ($n$) as $\vec{a}_i$

- Unit vectors (n-dim) $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

can write any n vector $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = b_1 \vec{e}_1 + \dots + b_n \vec{e}_n$

example : $\boxed{\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} = (-1)\vec{e}_1 + 3\vec{e}_2 + 5\vec{e}_3}$

$(-1)\vec{e}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} +$

$3\vec{e}_2 = + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$

# Linear combinations of Vectors

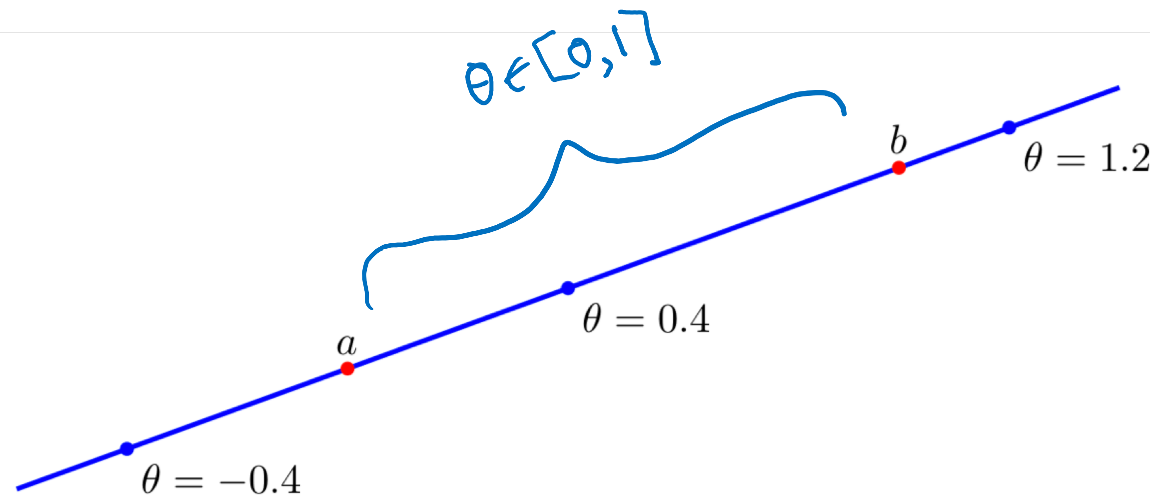



**Figure 1.12** The affine combination $(1-\theta)a + \theta b$ for different values of $\theta$. These points are on the line passing through $a$ and $b$; for $\theta$ between 0 and 1, the points are on the line segment between $a$ and $b$.

$\vec{c} = (1-\theta)\vec{a} + \theta\vec{b}$ , $0 \le \theta \le 1$

# Inner product

<u>Def</u>: (also dot product)

$\vec{a}, \vec{b}$ vectors of <u>same</u> dimension

$\left( \vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right)$ $\quad a^T b = \boxed{a_1 b_1 + \cdots + a_n b_n}$

scalar!

$\boxed{\langle a, b \rangle \text{ or } \langle a | b \rangle,\ (a, b)}$

ex: $\vec{a} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \langle a | b \rangle = ?$

<u>Properties</u>

- Commutative: $\vec{a}^T \vec{b} = \vec{b}^T \vec{a}$ or $\langle \vec{a} | \vec{b} \rangle = \langle \vec{b} | \vec{a} \rangle$ (vectors)
- associative with scalar multiplication $= (\gamma \vec{a})^T \vec{b} = \gamma (\vec{a}^T \vec{b})$

  $\gamma$ scalar
- distributive with vector addition

  $(\vec{a} + \vec{b})^T \vec{c} = \vec{a}^T \vec{c} + \vec{b}^T \vec{c}$

# Inner product

$(\vec{a}+\vec{b})^T(\vec{c}+\vec{d}) \overset{?}{=} \langle a|c\rangle + \langle a|d\rangle + \langle b|c\rangle + \langle b|d\rangle$

$\langle a+b|c+d\rangle$

examples

- unit vector? $\langle \vec{e}_i|\vec{a}\rangle = a_i$
- Sum: $\langle \frac{\vec{1}}{n}|\vec{a}\rangle = \frac{1}{n}(a_1+\dots+a_n)$
- square Norm (sum of squares): $\langle \vec{a}|\vec{a}\rangle = a_1^2+\dots+a_n^2$
- selective sum: $\vec{b} = (1,0,1,\dots,0)$

  $\langle \vec{a},\vec{b}\rangle =$ sum of $a_i$'s for all $i$ .st $b_i=1$
- Block vectors

  $\vec{a} = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_k \end{bmatrix}$, $\vec{b} = \begin{bmatrix} \vec{b}_1 \\ \vdots \\ \vec{b}_k \end{bmatrix}$, $\langle \vec{a},\vec{b}\rangle = \langle \vec{a}_1|\vec{b}_1\rangle + \dots + \langle \vec{a}_k|\vec{b}_k\rangle$

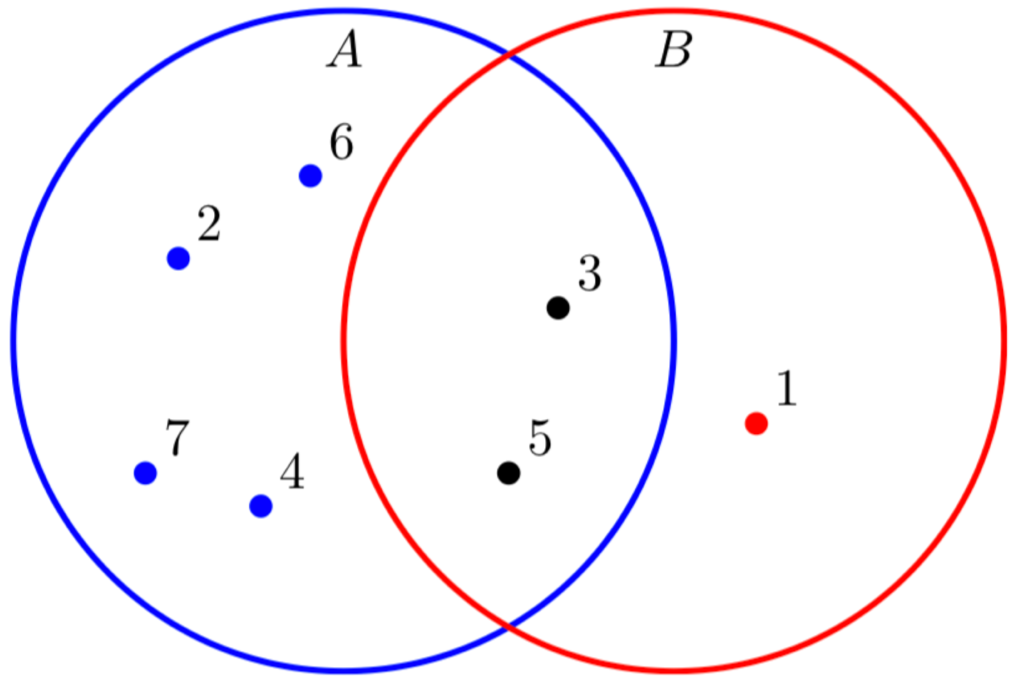
  dim is $m \cdot k$

# Inner product

$\vec{a} = (0,1,1,1,1,1,1)$

$\vec{b} = (1,0,1,0,1,0,0)$

$\langle \vec{a}, \vec{b} \rangle = 2$



# Inner product

- $\vec{f}$ set of features (age, income, ... )
  $\vec{w}$ vector of weights

  $\langle \vec{w} | \vec{f} \rangle$

- price-quantity:

  $\vec{p}$ prices, $\vec{q}$ quantities, $\langle \vec{p}, \vec{q} \rangle$ = total cost

- Polynomial evaluation

  $p(x) = c_1 + c_2 x + \dots + c_{n-1} x^{n-2} + c_n x^{n-1}$, $p(t) = \langle \vec{c} | \vec{z} \rangle$

  $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ vector of coefficients. $\vec{z} = \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{n-1} \end{bmatrix}$

# Complexity of vector computations

floating point = 64 bits OR 8 bytes
( a real number represent.)

Q: how many possible sequences of bits? $2^{64}$

Q: how many bytes to store n-vector?
$8n$

- floating point operation FLOP
- how many flops (as a function of dimension) do various operations take.

- $\beta\vec{a} \to n$ flops
- $\vec{x}+\vec{y} \to n$ flops

$\boxed{O(n)}$

$\langle x|y\rangle = x_1y_1 + \dots + x_ny_n$

n multiplications
n-1 additions

$\Big\}\ 2n-1$

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$\boxed{O(n)}$

sparse vectors ?

$nnz(\vec{x})$

- $a\vec{x}$ requires only $nnz(\vec{x})$ flops
- $\vec{x}+\vec{y}$ requires $\min\{nnz(\vec{x}), nnz(\vec{y})\}$ worst case

$(x+y)_i$

- $\langle \vec{x}, \vec{y} \rangle = 2\left[\min\{nnz(\vec{x}), nnz(\vec{y})\}\right]$ worst case

if $\vec{x}, \vec{y}$ have non-overlapping sparsity pattern, then ? 0 flops!

$\vec{x} = (1,1,1,0,0)$
$\vec{y} = (0,0,0,0,1)$

$\langle x|y \rangle = 1\cdot 1 + 1\cdot 1 + \cancel{1\cdot 0} + 0\cdot 0 + 0\cdot 0$