Linear Algebra

CSCI 2820

Lecture 21

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ECES 122

Today

- Introduction to determinants
- Invertibility and determinants
- Properties of Determinant
- Solving systems of equations

Two by two determinant

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $\underbrace{ad - bc}_{\text{determinant}} \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

"upper(lower) triangular matrix is invertible iff all diagonal entries are non-zero".

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{array}{l} \\ \times a_{11} \\ \times a_{11} \end{array} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \sim$$

$$\begin{array}{r} \\ \\ \times (a_{11}a_{22} - a_{12}a_{21}) \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \boxed{a_{11}a_{22} - a_{12}a_{21}} \neq 0 & a_{11}a_{23} - a_{13}a_{21} \\ 0 & \boxed{a_{11}a_{32} - a_{12}a_{31}} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix} - (a_{11}a_{32} - a_{12}a_{31}) \times \text{row } 2$$

Determinant

Compute $\det A$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$$

$$= 1 \cdot \det\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot \det\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \det\begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$= 1(0-2) - 5(0-0) + 0(-4-0) = \underline{-2}$$

$$\det\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} = \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix}$$

<u>notation</u>

<u>generally</u>: $A = [a_{ij}]$, define (i,j) co-factor C_{ij} of A

$C_{ij} = (-1)^{i+j} \det A_{ij}$. Then: $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$

(cofactor expansion across 1st row)

<u>i-th row</u>: $\det A = a_{i1}C_{i1} + \cdots + a_{in}C_{in}$

<u>j-th column</u>: $\det A = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj}$

$$\begin{bmatrix} + & - & + & \\ - & + & \cdot & \cdot \\ + & - & \vdots & \\ \cdot & & \ddots & \end{bmatrix}$$

Determinant

$$A \sim \begin{bmatrix} \overset{\neq 0}{a_{11}} & a_{12} & a_{12} \\ 0 & \overset{\neq 0}{a_{11}a_{22} - a_{12}a_{21}} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & \underline{a_{11}\Delta} \neq 0 \end{bmatrix}$$

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$$\boxed{A \text{ invertible iff } \Delta \neq 0}$$

Δ is determinant

$[A_{2\times 2}$ $\boxed{\det A = a_{11}a_{22} - a_{12}a_{21}}$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}]$

$$\Delta = (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) - (a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}) + (a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31})$$

$$= a_{11} \cdot \det A_{11} - a_{12} \det A_{12} + a_{13} \cdot \det A_{13}$$

$A_{11} = \begin{bmatrix} & A_{11} \end{bmatrix}$, $A_{12} = \begin{bmatrix} & \end{bmatrix}$, $A_{13} = \begin{bmatrix} A_{13} & \end{bmatrix}$

Determinant

let A square matrix

A_{ij} sub-matrix formed by deleting i-th row and j-th column from A

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

$$A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

[<u>note</u> $A = [a_{11}]$
$\det A = a_{11}$]

<u>Def</u>: For $n \geq 2$, determinant of $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$:

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\det A = 3 \cdot \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} + 0 \cdot C_{21} + 0 \cdot C_{31} + 0 \cdot C_{41} + 0 \cdot C_{51}$$

$$\det A = 3 \cdot 2 \det \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 3 \cdot 2(-2) = -12$$

<u>Thm</u>: If A is triangular, $\det A$ is product of the entries on the main diag.

$\det A_{2\times 2} = a_{11}a_{22} - a_{12}a_{21}$

Exercises

In Exercises 19–24, explore the effect of an elementary row operation on the determinant of a matrix. In each case, state the row operation and describe how it affects the determinant.

19. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$ad - cb$

$cb - ad$

flip a row: changes sign

Exercises

20. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$

$ad - cb$ $\quad (a+kc)d - c(b+kd)$

$= ad + \cancel{cdk} - cb - \cancel{cdk} =$ same.

if a multiple of a row is added to another row the det is same

Exercises

21. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$

$ad - cb$ $\quad akd - kcb = k(ad - cb)$

Exercises

22. $\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & 2 \\ 5+3k & 4+2k \end{bmatrix}$

det = 12 − 10 = 2

Same!

Exercises

23. $\begin{bmatrix} a & b & c \\ 3 & 2 & 1 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 \\ a & b & c \\ 4 & 5 & 6 \end{bmatrix}$

change sign

Exercises

24. $\begin{bmatrix} 1 & 0 & 1 \\ -3 & 4 & -4 \\ 2 & -3 & 1 \end{bmatrix}, \begin{bmatrix} k & 0 & k \\ -3 & 4 & -4 \\ 2 & -3 & 1 \end{bmatrix}$

$\times k$

Properties of Determinants

Thm 3: Row operations

A a square matrix

a. If a multiple of one row of A is added to another row to produce matrix B, det B = det A

b. If two rows of A are interchanged to produce B, det B = -det A

c. If one row of A is mult. by k to produce B, det B = k . det A

Properties of Determinants

Suppose square matrix A has been reduced to echelon form U by row replacements and row interchanges. there are r interchanges. by thm 3

$\det A = (-1)^r \cdot \det U$

$\det U = u_{11} \cdot \ldots \cdot u_{nn}$

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix}\text{product of} \\ \text{pivots in } U\end{pmatrix}, & A \text{ invertible} \\ 0, & A \text{ not invertible} \end{cases}$$

$\Downarrow$

$\boxed{\text{Thm: A square matrix } A \text{ is invertible iff } \det A \neq 0}$

$A = [\vec{a_1} \ldots \vec{a_n}]$, $\det A = \begin{cases} 0 & \text{not L.I} \\ \neq 0 & \text{L.I} \end{cases}$

$A = \begin{bmatrix} a & a & e \\ b & b & f \\ c & c & g \end{bmatrix}$, $\det A = 0$

- Column operations:

Thm

$$\boxed{\det A^T = \det A}$$

ex: induction use co-factor expansion 1st row (for A) or 1st column (for A^T).

- Multiplicative Property: A, B $n \times n$: $\det AB = (\det A) \cdot (\det B)$

$\det(A+B) \neq \det(A) + \det(B)$ generally ex: decide if

$\vec{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$

$$\det(\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3) = \begin{vmatrix} 5 & -3 & 2 \\ -7 & 3 & -7 \\ 9 & -5 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & -3 & 2 \\ -2 & 0 & -5 \\ 9 & -5 & 5 \end{vmatrix} = -(-3)\begin{vmatrix} -2 & -5 \\ 9 & 5 \end{vmatrix} - (-5)\begin{vmatrix} 5 & 2 \\ -2 & -5 \end{vmatrix}$$

$$= 3 \cdot 35 + 5(-21) = 0$$

so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dep.

<u>ex</u> : $A^2 = I$, show $\det A = \pm 1$

$\det I = 1$

$1 = \det I = \det A^2 = \det(A \cdot A) = (\det A) \cdot (\det A)$

$= (\det A)^2 \Rightarrow \det A = \pm 1$