Linear Algebra

CSCI 2820

Lecture 23

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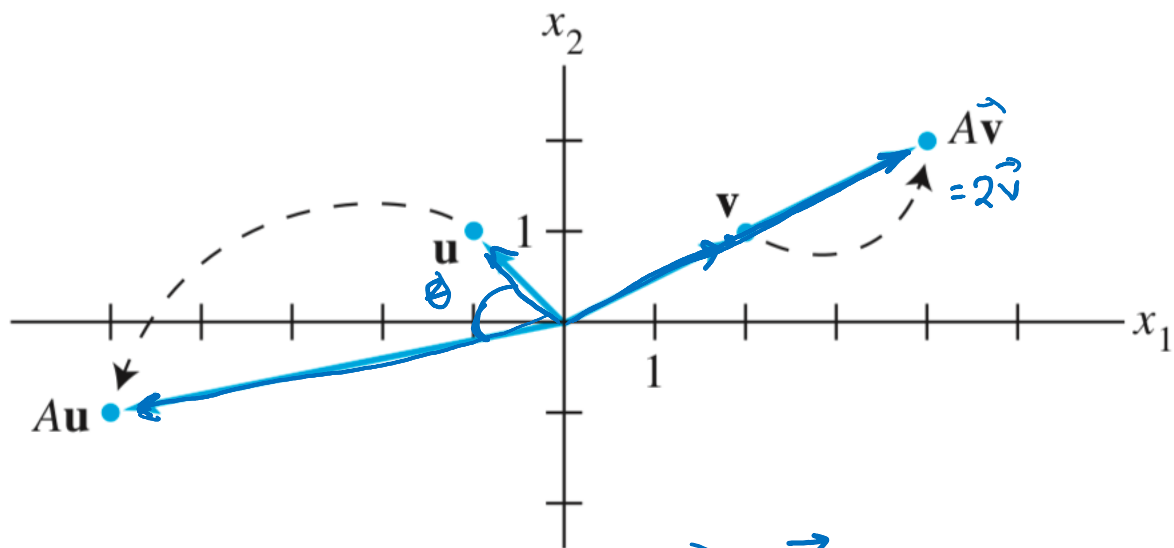
ECES 122

Today

- Eigenvalues
- Eigenvectors
- Characteristic Equations
- Determinant properties reminder
- Eigenvalue calculation

Eigenvalues and Eigenvectors



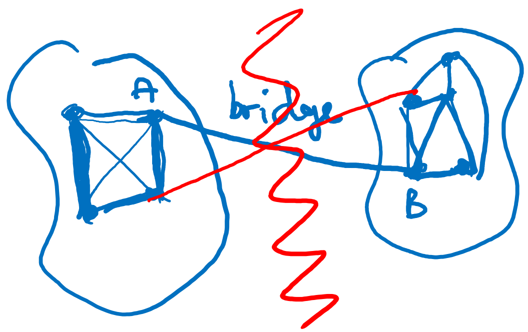
$A\vec{v} = \lambda \vec{v}$

$A : \vec{x} \longmapsto A\vec{x}$

$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

study equations: $A\vec{x} = 2\vec{x}$ or $A\vec{x} = -4\vec{x}$ ...

Eigenvalues and Eigenvectors



network
not well-connected
problem
fault-tolerance

$$A_G = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ \cdot & \cdot & \cdot \end{bmatrix}$$

Eigenvalues and Eigenvectors

Def: An eigenvector of an $n \times n$ matrix A is a non-zero vector $\vec{x}$ s.t $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a non-trivial solution $\vec{x}$ of $A\vec{x} = \lambda\vec{x}$ (such an $\vec{x}$ is called an eigenvector corresp. to eigenvalue λ)

eg: easy to determine. $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, \vec{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

are $\vec{u}, \vec{v}$ evectors?

$A\vec{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\vec{u}$ ✓

$A\vec{v} = \ldots = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ $\vec{v}$ not evector

Eigenvalues and Eigenvectors

eg. $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ show that $\lambda = 7$ is an evalue of A. Find corresponding evectors.

- $A\vec{x} = 7\vec{x}$ needs to have non-trivial soln.

$\boxed{(A - 7I)\vec{x} = \vec{0}} \Rightarrow \left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \right) \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \underbrace{\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}}_{\exists \text{ non-trivial sol.}} \vec{x} = \vec{0}$ How to find it?

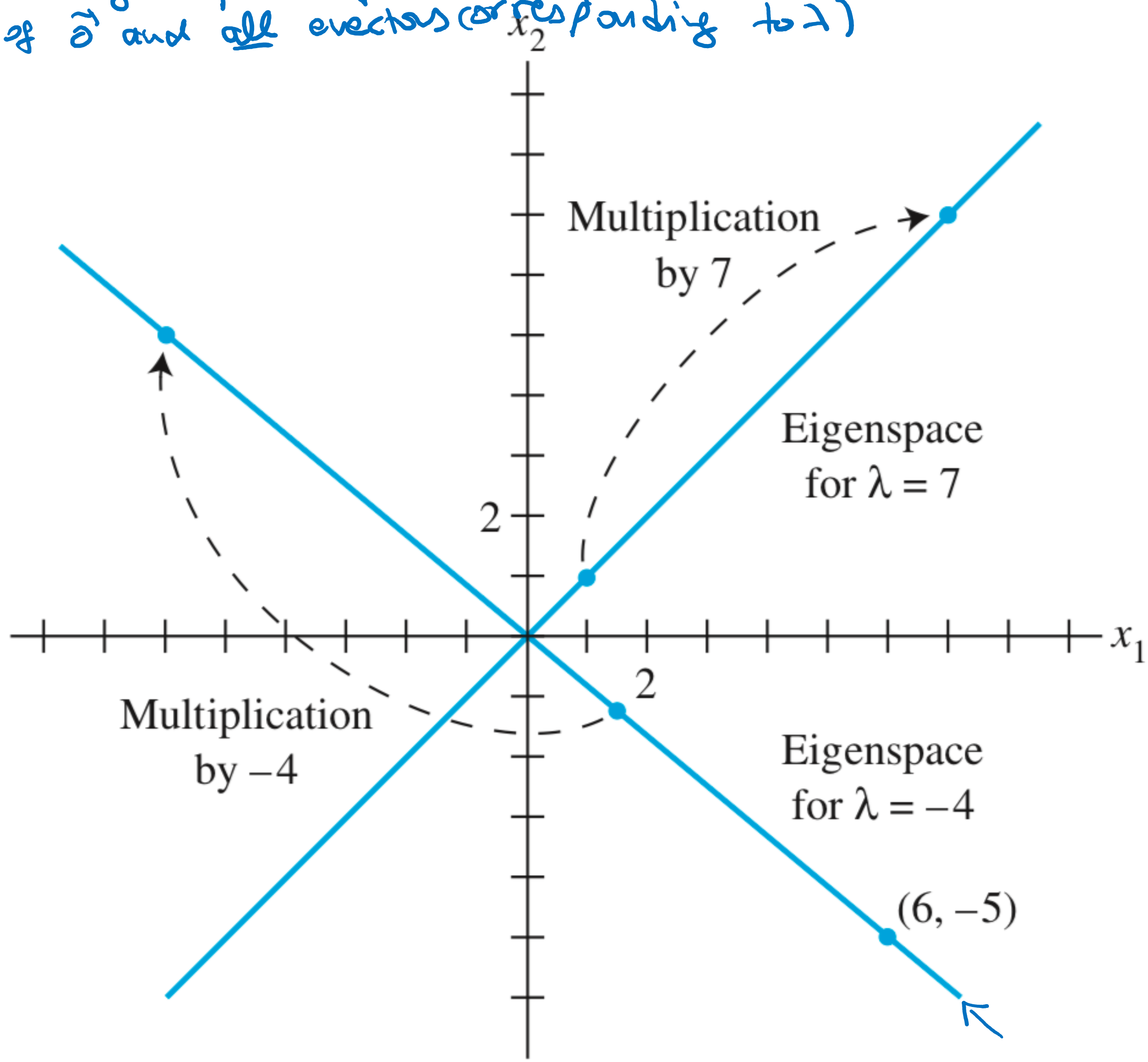
$\underset{A_1}{\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix}} \sim \underset{A_2}{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$ consistent,

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2 = c$

$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ evector of A of evalue 7.

- For any λ: $(A - \lambda I)\vec{x} = \vec{0}$ does have non-trivial sol. $\vec{x}$ $\Leftrightarrow$ Does matrix A have evalue λ

Set of all solutions of (*) is null$(A-\lambda I)$. It is a subspace of $\mathbb{R}^n$
called "eigenspace" of A corresponding to λ.
(consists of $\vec{0}$ and __all__ evectors corresponding to λ)



Eigenvalues and Eigenvectors

eg $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. an evalue of A is 2.

Find a basis for the corresp. eigenspace.

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}. \quad (A-2I)\vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

indeed consistent
2 is evalue

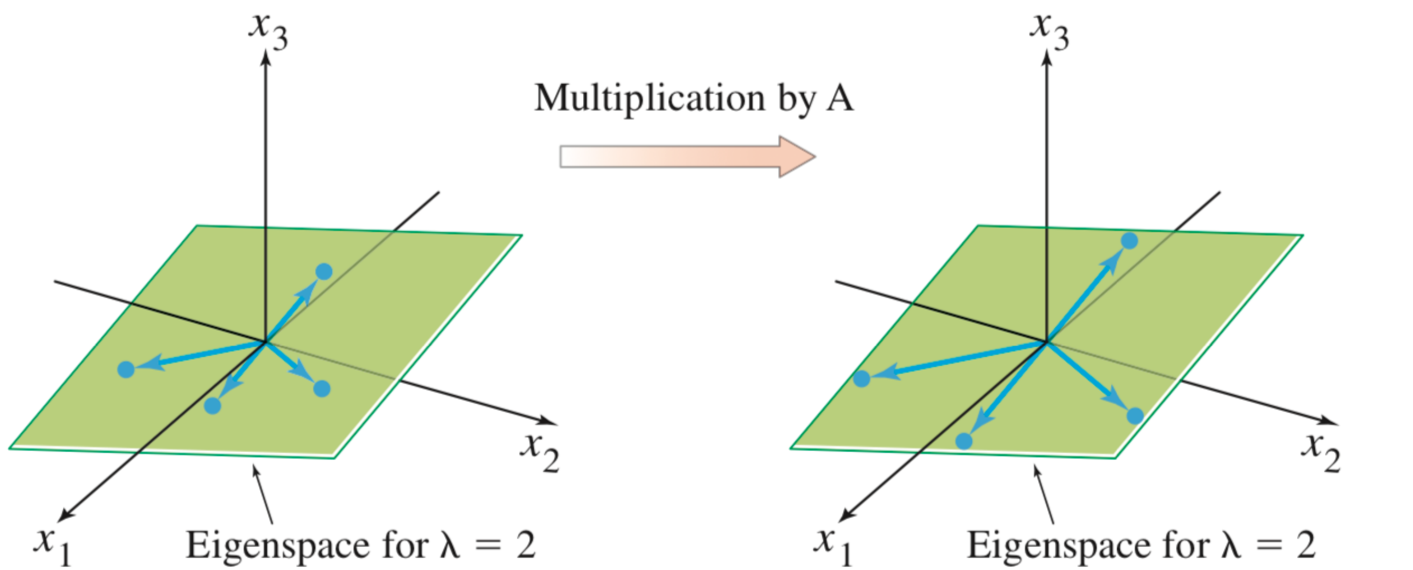
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$2x_1 - x_2 + 6x_3 = 0$

$x_1 = \frac{x_2}{2} - 3x_3$

basis for 2-d eigenspace: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Eigenvalues and Eigenvectors



Eigenvalues and Eigenvectors

Thm 1: The eigenvalues of a triangular matrix are the entries on its main diag.

Proof: $A_{3\times 3}$, upper-triangular

$\boxed{(A-\lambda I)\vec{x}=\vec{0}}$ non-trivial sol, for what λ?

$$A-\lambda I=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}-\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$=\begin{bmatrix} \underline{a_{11}-\lambda} & a_{12} & a_{13} \\ 0 & \underline{a_{22}-\lambda} & a_{23} \\ 0 & 0 & \underline{a_{33}-\lambda} \end{bmatrix} \leftarrow$$

If $A-\lambda I$ invertible, then unique sol is $(A-\lambda I)^{-1}\vec{0}=\vec{0}$

so $A-\lambda I$ not invertible for λ to be evalue.

$\boxed{\lambda=a_{11} \text{ or } \lambda=a_{22} \text{ or } \lambda=a_{33}}$

Eigenvalues and Eigenvectors

what does it mean for A to have evalue 0?

eg. $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & \boxed{0} & 6 \\ 0 & 0 & 2 \end{bmatrix}$ $\lambda_1 = 3$, $\lambda_2 = 0$, $\lambda_3 = 2$

$A\vec{x} = 0\vec{x} \Rightarrow A\vec{x} = \vec{0}$ has non-trivial sol.

$\Rightarrow$ iff A is not invertible.

"0 is evalue of A iff A is not invert"

<u>Thm</u>: If $\vec{v}_1, \ldots, \vec{v}_r$ eigenvectors of A correspond to distinct evalues $\lambda_1, \ldots, \lambda_r$ of $n \times n$ matrix, then the set $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly indep.

<u>Proof</u>: Towards $\Rightarrow\Leftarrow$, suppose $\{v_1, \ldots, v_r\}$ is linearly dependent.

Let p be the index s.t. $\boxed{\vec{v}_{p+1} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p}$ (1) $\quad A\vec{v}_k = \lambda_k \vec{v}_k$

$c_1 A\vec{v}_1 + \cdots + c_p A\vec{v}_p = A\vec{v}_{p+1}$

$\Rightarrow \boxed{c_1\lambda_1\vec{v}_1 + \cdots + c_p\lambda_p\vec{v}_p = \lambda_{p+1}\vec{v}_{p+1}}$ (2)

mult (1) by λ_{p+1}:

$\boxed{c_1\lambda_{p+1}\vec{v}_1 + \cdots + c_p\lambda_{p+1}\vec{v}_p = \lambda_{p+1}\vec{v}_{p+1}}$ (3)

Eigenvalues and Eigenvectors

subtract (2) - (3) :

$$\boxed{c_1(\lambda_1 - \lambda_{p+1})\vec{v}_1 + \dots + c_p(\lambda_p - \lambda_{p+1})\vec{v}_p = \vec{0}} \quad (4)$$

$\{\vec{v}_1, \dots, \vec{v}_p\}$ L.I : $c_i(\lambda_i - \lambda_{p+1}) = 0 \quad \forall i = 1 \dots p$

$\neq 0$

$\boxed{c_i = 0} \Rightarrow \vec{v}_{p+1} = \vec{0}$

hence $\{v_1, \dots, v_r\}$ are l.ind. $\Rightarrow\!\!\!\!\times$

Q1 : If $\vec{x}$ is evector of A corresp to λ,

what is $A^3\vec{x}$?

$$\lambda^3\vec{x} = A^2(A\vec{x}) = A^2\lambda\vec{x} = \lambda A^2\vec{x} = \lambda A(A\vec{x}) = \lambda^2 A\vec{x} = \lambda^2\lambda\vec{x} = \lambda^3\vec{x},$$

in general : $A^k\vec{x} = \lambda^k\vec{x}$

~~Characteristic equations~~

Q2: Suppose $\vec{b}_1, \vec{b}_2$ are evectors corresponding to distinct evalues λ_1, λ_2 and suppose $\vec{b}_3, \vec{b}_4$ are l.ind. evectors corresponding to a third distinct evalue λ_3.

$\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\}$ l.ind?

suppose: $\boxed{c_1\vec{b}_1 + c_2\vec{b}_2 + \underbrace{(c_3\vec{b}_3 + c_4\vec{b}_4)}_{}= \vec{0}}$ ∗

$\vec{0}$

$\vec{0}$

$= \vec{v}$, in eigenspace of λ_3 → $c_3\vec{b}_3 + c_4\vec{b}_4 = \vec{0}$

→ $c_3\vec{b}_3 + c_4\vec{b}_4$ is eigenvector of λ_3

- if $c_3\vec{b}_3 + c_4\vec{b}_4$ was evector for λ_3, then $\{\vec{b}_1, \vec{b}_2, c_3\vec{b}_3 + c_4\vec{b}_4\}$ l.indep $\Rightarrow$ $c_1 = c_2 = 0$ and $c_3\vec{b}_3 + c_4\vec{b}_4 = \vec{0}$ $\Rightarrow\Leftarrow$
- $c_3\vec{b}_3 + c_4\vec{b}_4 = \vec{0}$ $\Rightarrow$ $c_1\vec{b}_1 + c_2\vec{b}_2 = \vec{0}$, know $\{\vec{b}_1, \vec{b}_2\}$ l.ind.

$c_1 = c_2 = 0$. Also → $c_3 = c_4 = 0$

Characteristic equations

Characteristic equations

Characteristic equations

Similarity

Similarity