Linear Algebra

CSCI 2820

Lecture 3

Prof. Alexandra Kolla

Alexandra.Kolla@Colorado.edu
ECES 122

Today

- Linear functions
- Inner products and linear functions
- Examples
- Taylor series (if we have time)

Refresher on Linear combinations

Geometrically:

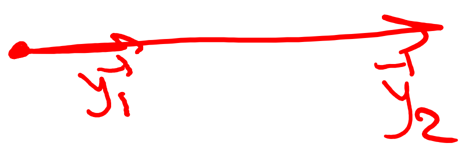
- (1) what ~~is~~ are the the linear combinations of a nonzero vector (1D)?
- (2) the linear combinations of two nonzero vectors? (2D)

  $\vec{y} = a_1\vec{u} + a_2\vec{v}$, $a_i \in \mathbb{R}$

  $\boxed{\vec{y}_1 = 2\vec{u} + 1.5\vec{v}}$

  $\boxed{\vec{y}_2 = 200\vec{u} + 100\vec{v} - 1.1\vec{u} + 5\vec{v}}$
- When are (1) and (2) the same?

  $\vec{y}_1 = 2.5\vec{v}$, $\vec{y}_2 = 403.9\vec{v}$
- Example of linear independence!



Refresher on Linear combinations

Geometrically:

- Write $\begin{pmatrix} 1.4 \\ 2 \\ -4.3 \end{pmatrix}$ as a linear combinations of 3 unit vectors. e_1, e_2, e_3
- Generally, we can choose what we call a basis and write any vector as a linear combination of basis vectors.
- Solving systems of linear equations?

Linear Functions

Function notation : $f : \underbrace{\mathbb{R}^n}_{\text{domain}} \to \underbrace{\mathbb{R}}_{\text{range}}$

$f(\vec{x})$ = some scalar $= f(x_1, \dots, x_n)$

$\vec{x} = (x_1, \dots, x_n)$

e.g. $f : \underbrace{\mathbb{R}^4}\to \underbrace{\mathbb{R}}$, $f_0(\vec{x}) = x_1 + x_2 - x_4^2$

$\to f_1(\vec{x}) = 5$ constant function

$f : \mathbb{R}^{10} \to \mathbb{R}$ $\to f_2(\vec{x}) = 5$

$f_1((0,1,1,-5)) = 5$

$f_1((0,1,1,1,0,0,0,0,1,-5))$ doesn't make sense

Sum function : input vector, output sum of coordinates

Linear Functions

Inner product function: let $\vec{a}$ be an n-vector

$$f_a(\vec{x}) = \vec{a}^T\vec{x}$$
$$(\langle \vec{a} | \vec{x} \rangle)$$

- Linear functions: superposition property

$$\boxed{f(\gamma\vec{x}+\beta\vec{y}) = \gamma f(\vec{x}) + \beta f(\vec{y})} \quad [1]$$

$$\begin{aligned} f_a(\gamma\vec{x}+\beta\vec{y}) &= \langle \vec{a} | \gamma\vec{x}+\beta\vec{y} \rangle \\ &= \langle \vec{a} | \gamma\vec{x} \rangle + \langle \vec{a} | \beta\vec{y} \rangle \\ &= \gamma \underbrace{\langle \vec{a} | \vec{x} \rangle} + \beta \underbrace{\langle \vec{a} | \vec{y} \rangle} \\ &= \gamma f_a(\vec{x}) + \beta f_a(\vec{y}) \end{aligned}$$

inner product function $\longrightarrow$ superposition

Def: Linear function f iff it satisfies [1]

Linear Functions as Inner Products

any linear f: $f(\underbrace{a_1\vec{x}_1} + \underbrace{\cdots + a_k\vec{x}_k}) = a_1 f(\vec{x}_1) + \cdots + a_k f(\vec{x}_k)$

$= f(a_1\vec{x}_1 + \quad) = a_1 f(\vec{x}_1) + f(\underbrace{a_2\vec{x}_2} + \underbrace{\cdots + a_k\vec{x}_k})$

$= a_1 f(\vec{x}_1) + a_2 f(\vec{x}_2) + f(a_3\vec{x}_3 + \cdots + a_k\vec{x}_k)$

$\vdots$

$= a_1 f(\vec{x}_1) + \cdots + a_k f(\vec{x}_k)$

- Homogeneity: $f(a\vec{x}) = a f(\vec{x})$ [2]
- Additivity: $f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$ [3]

$\Updownarrow$

superposition or eq[1]

already saw: Inner product function is linear

we will see: If a function is linear, then it can always be expressed as an inner product of its argument with some fixed vector

Affine Functions

$f: \mathbb{R}^n \to \mathbb{R}$

assume f is linear (superposition is satisfied)

$$f(\vec{x}) = f(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) \overset{[1]}{=} x_1 f(\vec{e}_1) + \cdots + x_n f(\vec{e}_n)$$

$$= \langle \vec{a}, \vec{x} \rangle$$

$$\to \vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$

define $\underbrace{\vec{a} = (f(\vec{e}_1), \ldots, f(\vec{e}_n))}$

unique $\vec{a}$ s.t. a linear function can be written as $\langle \vec{a}, \vec{x} \rangle$

Pf.: Assume, towards contradiction, that

$f(\vec{x}) = \langle \vec{a}, \vec{x} \rangle$ and $f(\vec{x}) = \langle \vec{b}, \vec{x} \rangle$

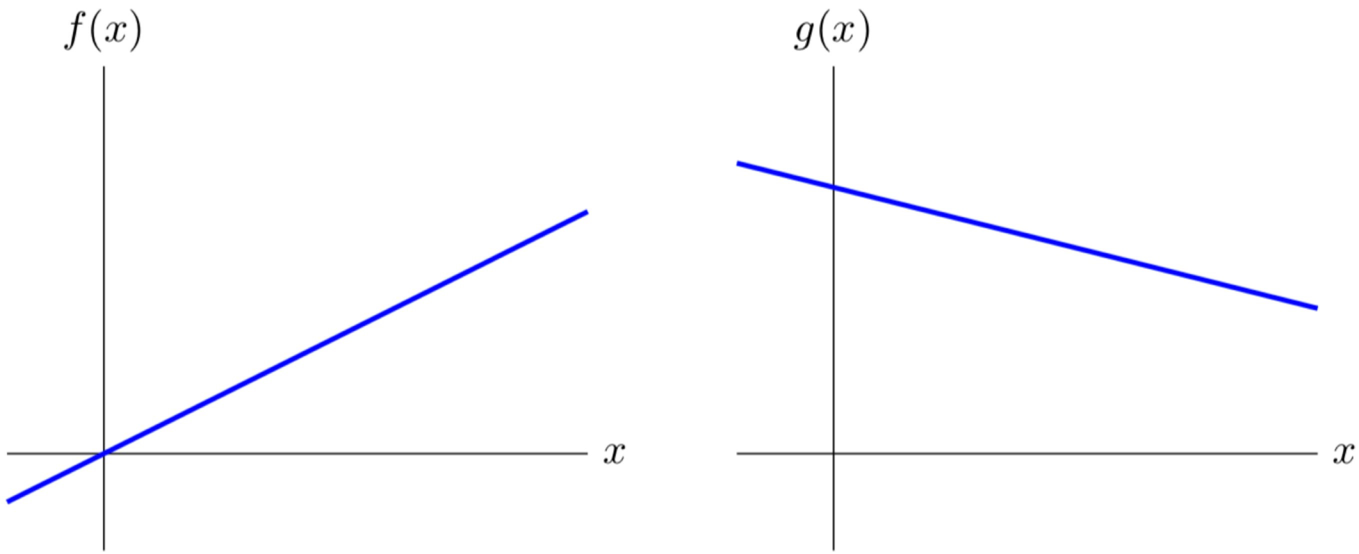
take $\vec{x} = \vec{e}_i$: $f(\vec{e}_i) = a_i$ and also $f(\vec{e}_i) = b_i$

$a_i = b_i \quad \forall i \quad \Rightarrow \vec{a} = \vec{b}$

Examples: 2D

counterexample $(n=2)$: $\vec{x} = (1,-1)$, $\vec{y} = (-1,1)$, $\alpha = \beta = 1/2$

$$\boxed{\begin{aligned} & f(\alpha\vec{x}+\beta\vec{y}) \\ \neq\ & \alpha f(\vec{x}) + \beta f(\vec{y}) \end{aligned}} \quad \begin{aligned} &= f(\vec{0}) = 0 \\ &= \tfrac{1}{2}\max\{1,-1\} + 1/2\max\{1,-1\} = 1 \neq 0 \end{aligned}$$



ex1: Average of n-vector: $f(\vec{x}) = (x_1 + \dots + x_n)/n$

$f(\vec{x}) = \langle \vec{a}, \vec{x} \rangle$ $\quad \vec{a} = (1/n, \dots, 1/n) = \frac{\vec{1}}{n}$

ex2: Maximum element of n-vector $\vec{x}$, $f(\vec{x}) = \max\{x_1, \dots, x_n\}$

Affine Functions

f is affine : $f(\vec{x}) = \langle \vec{a}, \vec{x} \rangle + b$

$f : \mathbb{R}^n \to \mathbb{R}$

ex. $f(\vec{x}) = \underbrace{2.3}_{\substack{\text{scalar} \\ b}} \underbrace{- 2x_1 + 1.3x_2 - x_3}_{\substack{\text{linear} \\ \vec{a} = (-2, 1.3, -1)}}$

$$\boxed{\begin{array}{l}\text{variation of superposition:} \\ f(\alpha\vec{x} + \beta\vec{y}) = \alpha f(\vec{x}) + \beta f(\vec{y}) \\ \quad \text{for } \alpha + \beta = 1\end{array}}$$ [4]

take $\gamma, \beta : \gamma + \beta = 1$: $f(\gamma\vec{x} + \beta\vec{y}) = \langle \vec{a} | \gamma\vec{x} + \beta\vec{y} \rangle + b$

$= \gamma \langle \vec{a} | \vec{x} \rangle + \beta \langle \vec{a} | \vec{x} \rangle + \underline{(\gamma + \beta)} b$

$= \gamma f(\vec{x}) + \beta f(\vec{y})$

- Any $f: \mathbb{R}^n \to \mathbb{R}$ that satisfies restricted superposition property [4] is affine.

$$\boxed{f(\vec{x}) = f(\vec{0}) + x_1\left(f(\vec{e}_1) - f(\vec{0})\right) + \cdots + x_n\left(f(\vec{e}_n) - f(\vec{0})\right)}$$

[exercise to show that every affine function can be written like this]

① assume $f(\vec{x}) = \langle \vec{a}, \vec{x} \rangle + b$

② assume only restricted superposition

Example