Linear Algebra

CSCI 2820

Lecture 6

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ECES 122

Today

- Cauchy-Schwartz
- Angles
- Complexity
- Examples

Refresher on Distance/std

For a vector $\vec{v}$, let $\vec{u} = \dfrac{\tilde{v}}{std(\vec{v})}$ what is the 2-norm of $\vec{u}$?

$$\|\vec{u}\| = ?$$

$$\|\vec{u}\| = \frac{\|\tilde{v}\|}{\|std(\vec{v})\|} = \frac{\|\tilde{v}\|}{\frac{\|\tilde{v}\|}{\sqrt{n}}} = \sqrt{n}$$

$$\mathbf{std}(x) = \frac{\|x - (\mathbf{1}^T x/n)\mathbf{1}\|}{\sqrt{n}}.$$

Cauchy-Schwartz Inequality

C-S Ineq: $|\langle a,b\rangle| \le \|\vec{a}\|\cdot\|\vec{b}\|$ for any n-vectors $\vec{a}, \vec{b}$

$$\underbrace{|a_1b_1 + \cdots + a_nb_1|}_{|\sum a_ib_i|} \le \underbrace{(a_1^2 + \cdots + a_n^2)^{1/2}}_{\sqrt{\sum a_i^2}} \underbrace{(b_1^2 + \cdots + b_n^2)^{1/2}}_{\sqrt{\sum b_i^2}}$$

$$|\sum a_ib_i| \le \sqrt{\sum a_i^2} \cdot \sqrt{\sum b_i^2}$$

Proof:

if $\vec{a}$ or $\vec{b} = \vec{0}$, then immediate

so assume $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$ (so $\|\vec{a}\|, \|\vec{b}\| \neq 0$)

define $\gamma = \|\vec{a}\|$, $\beta = \|\vec{b}\|$.

$\boxed{\|\beta\vec{a} - \gamma\vec{b}\|^2} \geq 0 \Rightarrow (\beta\vec{a} - \gamma\vec{b})^T(\beta\vec{a} - \gamma\vec{b}) \geq 0$

$\Rightarrow \beta^2\|\vec{a}\|^2 - 2\beta\gamma\langle\vec{a},\vec{b}\rangle + \gamma^2\|\vec{b}\|^2 \geq 0$

$\Rightarrow \|\vec{b}\|^2\|\vec{a}\|^2 - 2\|\vec{b}\|\|\vec{a}\|\langle\vec{a},\vec{b}\rangle + \|\vec{a}\|^2\|\vec{b}\|^2 \geq 0$

$\Rightarrow 2\|\vec{a}\|^2\|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\langle\vec{a},\vec{b}\rangle \geq 0$ divide by $2\|\vec{a}\|\|\vec{b}\|$

$\Rightarrow \langle\vec{a},\vec{b}\rangle \le \|\vec{a}\|\|\vec{b}\|$, apply this to $-\vec{a}$

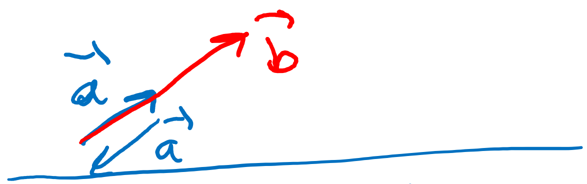
Cauchy-Schwartz Inequality

when is it tight? (equality)

tight $\|\beta\vec{a} - \gamma\vec{b}\| = 0$, $\beta\vec{a} = \gamma\vec{b}$ $(\beta,\gamma \neq 0)$

if $\vec{a}$ is a scalar multiple of $\vec{b}$ (or vice versa)

$$|\langle\vec{a},\vec{b}\rangle| \le \|\vec{a}\|\|\vec{b}\|$$



verify triangle inequality:

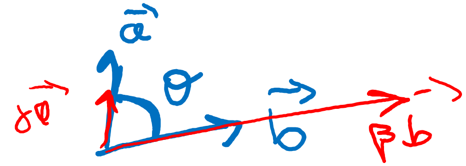
$$\left(\|\vec{a}+\vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|\right) (*)$$

$$\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b})^T(\vec{a}+\vec{b}) = \|\vec{a}\|^2 + 2\langle\vec{a},\vec{b}\rangle + \|\vec{b}\|^2$$ (cauchy-schwartz)

$$\le \|\vec{a}\|^2 + 2\|\vec{a}\|\|\vec{b}\| + \|\vec{b}\|^2 = (\|\vec{a}\|+\|\vec{b}\|)^2$$

take sqrt, get (*)

Angles



$$\theta = \arccos\left(\frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\|\|\vec{b}\|}\right),$$

$$\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$\|\vec{a}\| = \|\vec{b}\| = 1$$

angle between $\vec{a}, \vec{b}$:

$$\angle(\vec{a}, \vec{b})$$

they $\cos\theta = \langle \vec{a}, \vec{b} \rangle$

$\pi/2$ is 90°

$\pi/3$ is 60°

- $\angle(\vec{a}, \vec{b}) = \angle(\vec{b}, \vec{a})$
- $\angle(\gamma\vec{a}, \beta\vec{b}) = \angle(\vec{a}, \vec{b}) \ \forall \gamma, \beta > 0$

- classify angles according to the sign of $\langle \vec{a}, \vec{b} \rangle$

$\rightarrow$ if $\angle(\vec{a}, \vec{b}) = 90^\circ$, $\langle \vec{a}, \vec{b} \rangle = 0$ orthogonal, $\vec{a} \perp \vec{b}$

$\rightarrow$ if $\angle(\vec{a}, \vec{b}) = 0$, then $\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\|\|\vec{b}\|$ aligned

$\vec{a} = |\gamma| \cdot \vec{b}$

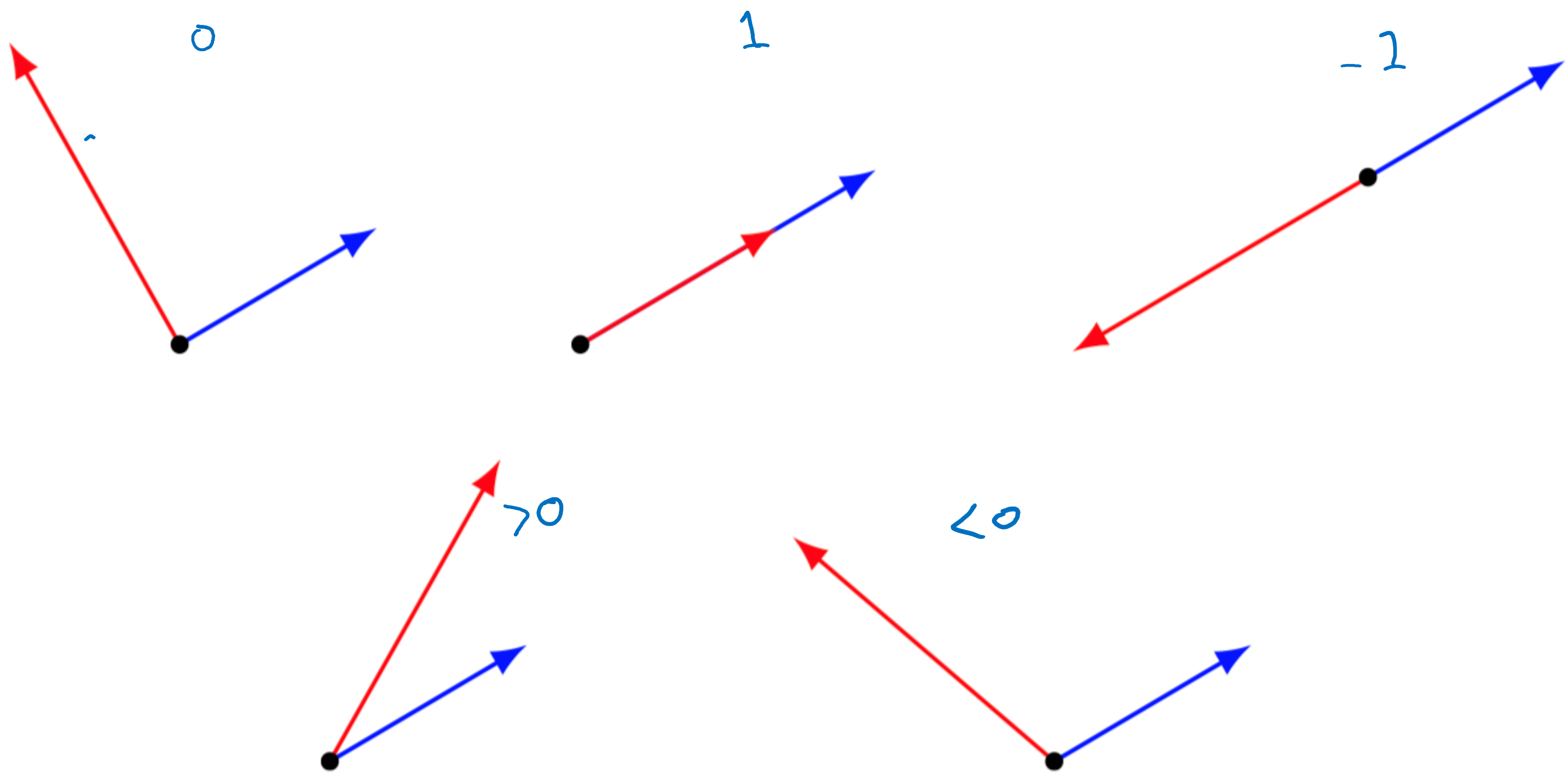
$\rightarrow$ if $\angle(\vec{a}, \vec{b}) = 180^\circ$, then $\langle \vec{a}, \vec{b} \rangle = -\|\vec{a}\|\|\vec{b}\|$, anti-aligned

$\vec{a} = -|\gamma||\vec{b}|$

Angles

- $\angle(\vec{a},\vec{b}) < 90^\circ$ $(\pi/2)$, acute angle $\Leftrightarrow \langle \vec{a},\vec{b} \rangle > 0$
- $\angle(\vec{a},\vec{b}) > 90^\circ$, obtuse angle $\Leftrightarrow \langle \vec{a},\vec{b} \rangle < 0$

Angles, examples



Angles, examples

- sperical distance

  $R \cdot \angle(\vec{a}, \vec{b})$

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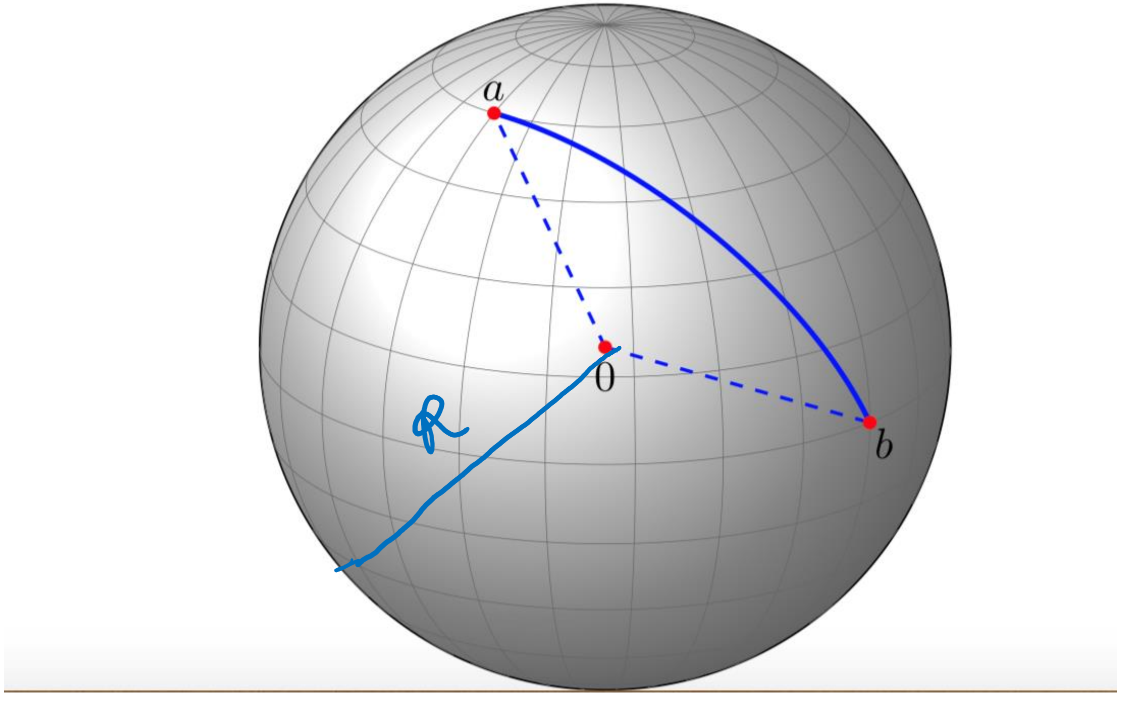


Angles, examples

	Veterans Day	Memorial Day	Academy Awards	Golden Globe Awards	Super Bowl
Veterans Day	0	60.6	85.7	87.0	87.7
Memorial Day	60.6	0	85.6	87.5	87.5
Academy A.	85.7	85.6	0	58.7	85.7
Golden Globe A.	87.0	87.5	58.7	0	86.0
Super Bowl	87.7	87.5	86.1	86.0	0

$\vec{x}, \vec{y} \Rightarrow$ word counts for two documents $\rightarrow$

$\angle(\vec{x}, \vec{y})$ to measure similarity

Can the angle be more than 90 degrees?

Angles

Norm of the sum of two vectors $\vec{x}, \vec{y}$

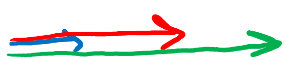
$$\|\vec{x}+\vec{y}\|^2 = (\vec{x}+\vec{y})^T(\vec{x}+\vec{y}) = \|\vec{x}\|^2 + 2\langle\vec{x},\vec{y}\rangle + \|\vec{y}\|^2$$

$$= \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\|\underline{\cos\theta} + \|\vec{y}\|^2$$

$\theta = \angle(\vec{x},\vec{y})$

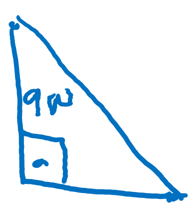
- if $\vec{x},\vec{y}$ are aligned: $\|\vec{x}+\vec{y}\|^2 = (\|\vec{x}\|+\|\vec{y}\|)^2$

  $\theta = 0$, $\cos\theta = 1$ $\qquad \|\vec{x}+\vec{y}\| = \|\vec{x}\|+\|\vec{y}\|$



- if $\vec{x},\vec{y}$ are orthogonal ($\theta = 90°$) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

  $\cos\theta = 0$



Pythagorean theorem

Correlation coefficient

$\vec{a}, \vec{b}$

$\tilde{a} = \vec{a} - \text{avg}(\vec{a})\vec{1}$

$\tilde{b} = \vec{b} - \text{avg}(\vec{b})\vec{1}$

correlation coefficient

$$\boxed{\rho = \frac{\langle \tilde{a}, \tilde{b} \rangle}{\|\tilde{a}\| \|\tilde{b}\|}}, \quad \rho = \cos\theta$$

$$\theta = \angle(\tilde{a}, \tilde{b})$$

also $\vec{u} = \dfrac{\tilde{a}}{\text{std}(\vec{a})}$, $\vec{v} = \dfrac{\tilde{b}}{\text{std}(\vec{b})}$, $\boxed{\rho = \dfrac{\langle \vec{u}, \vec{v} \rangle}{n}}$ $[\|u\| = \|v\| = \sqrt{n}]$

- $\rho = 0$ uncorrelated when $\tilde{a}, \tilde{b}$ are orthogonal

Correlation coefficient

highly correlated

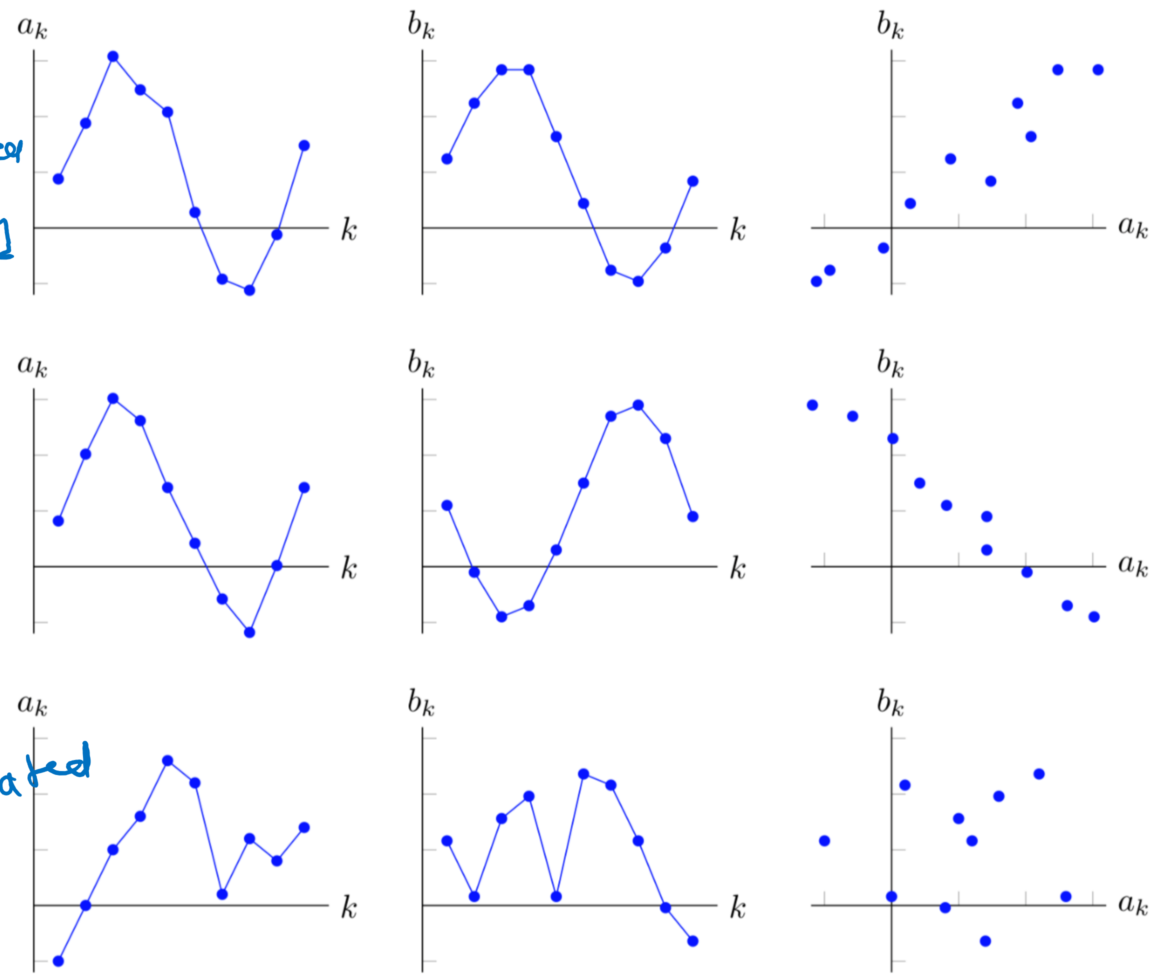
ρ close to 1

anti-cor

ρ close to −1

uncorrelated

ρ near 0



Std of sum

claim: $\boxed{std(\vec{a}+\vec{b})^2 = std(\vec{a})^2 + 2\rho\, std(\vec{a})\, std(\vec{b}) + std(\vec{b})^2}$

$\tilde{a}, \tilde{b}$ de-meaned vectors of $\vec{a}, \vec{b}$

$\tilde{a}+\tilde{b}$ de-meaned vector of $\vec{a}+\vec{b}$

def of std: $std(\vec{a}+\vec{b})^2 = \dfrac{\|\tilde{a}+\tilde{b}\|^2}{n}$, using norm of sum

$$n.\,std(a+b)^2 = \|\tilde{a}+\tilde{b}\|^2 = \|\tilde{a}\|^2 + 2\|\tilde{a}\|\|\tilde{b}\|\cos\theta + \|\tilde{b}\|^2$$

$$= \|\tilde{a}\|^2 + 2\rho\|\tilde{a}\|\|\tilde{b}\| + \|\tilde{b}\|^2$$

$$= n\,std(\tilde{a})^2 + 2\rho n\, std(\tilde{a})\, std(\tilde{b}) + n\,std(b)^2$$

- $\rho = 1$: $std(\vec{a}+\vec{b}) = std(\vec{a}) + std(\vec{b})$ (1) $\qquad (2) < (1)$
- $\rho = 0$: $std(\vec{a}+\vec{b}) = \sqrt{std^2(\vec{a}) + std(\vec{b})^2}$ (2)
- $\rho = -1$: $std(\vec{a}+\vec{b}) = |std(\vec{a}) - std(\vec{b})|$

Complexity