Linear Algebra

CSCI 2820

Lecture 7

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ECES 122

Today

- Linear Independence of vectors
- Bases
- Vector Spaces

Refresher on Angles

When does the triangle inequality hold with equality?

i.e. what are conditions of vectors $\vec{a}, \vec{b}$ so that

$$\left|\left|\vec{a} + \vec{b}\right|\right| = ||\vec{a}|| + \left|\left|\vec{b}\right|\right|$$

$\theta = 0$

$\vec{a}$ $\vec{b}$

Linear Independence

Def: A collection of vectors $\vec{a}_1, \dots, \vec{a}_k$ is called linearly dependent if (L.D)

$$\boxed{\beta_1 \vec{a}_1 + \cdots + \beta_k \vec{a}_k = \vec{0}}$$

holds for some $\beta_1, \dots, \beta_k$ that are not all zero.

if $\vec{a}_1, \dots, \vec{a}_k$ L.D then at least one of them can be written as a linear comb. of the others.

Proof: assume $\beta_i \neq 0$:

$$\beta_i \vec{a}_i = -\beta_1 \vec{a}_1 - \cdots - \beta_k \vec{a}_k$$

$$\vec{a}_i = \left(-\beta_1/\beta_i\right) \vec{a}_1 + \cdots + \left(-\frac{\beta_k}{\beta_i}\right) \vec{a}_k$$

Converse also true: if $\vec{a}_i = \sum c_j \vec{a}_j$ then $\{\vec{a}_1, \dots, \vec{a}_k\}$ L.D

Linear Independence

Def: Linear Independent vectors

A collection of n-vectors $\vec{a}_1, \dots, \vec{a}_k$ $(k \geq 1)$

is linearly independent:

$$\beta_1 \vec{a}_1 + \dots + \beta_k \vec{a}_k = \vec{0}$$ only holds

for $\beta_1 = \dots = \beta_k = 0$.

E.g: ① $\{\vec{a}\}$: when is this linearly dep?

if $\beta \neq 0$ $\beta \vec{a} = \vec{0}$ then L.D, $\vec{a} = \vec{0}$

$\vec{a} \neq \vec{0}$ linearly indep.

② Any list containing the zero vector is L.D

$\beta \neq 0$ $\beta \cdot \vec{0} + \sum \beta_i \vec{a}_i = \vec{0}$, set $\beta_i = 0$

③ $\{\vec{a}_1, \vec{a}_2\}$ when are they L.D? $\vec{a}_1 = \beta \cdot \vec{a}_2 \Rightarrow \vec{a}_1 + (-\beta)\vec{a}_2 = \vec{0}$

Linear Independence

(4) $\vec{a}_1 = \begin{bmatrix} 0.2 \\ -7 \\ 8.6 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} -0.1 \\ 2.0 \\ -1.0 \end{bmatrix}$, $\vec{a}_3 = \begin{bmatrix} 0.0 \\ -1.0 \\ 2.2 \end{bmatrix}$

$\vec{a}_1 + 2\vec{a}_2 - 3\vec{a}_3 = \vec{0} \Rightarrow \vec{a}_2 = (-1/2)\vec{a}_1 + (\frac{3}{2})\vec{a}_3$

(5) $\vec{e}_1, \ldots, \vec{e}_n$ linearly indep.

assume $\beta_1\vec{e}_1 + \cdots + \beta_n\vec{e}_n = \vec{0} \Rightarrow$

$$\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = \vec{0} \Rightarrow \beta_i = 0$$

<u>Linear combination of Linearly Indep vectors:</u>

$\vec{x} = \beta_1\vec{a}_1 + \cdots + \beta_k\vec{a}_k$, $\{\vec{a}_1, \ldots, \vec{a}_k\}$ is l. indep

<u>Claim</u>: coefficients that form $\vec{x}$ are <u>unique</u>

<u>Proof</u>: assume, toward $\rightarrow\leftarrow$, that we also have $\vec{x} = \gamma_1\vec{a}_1 + \cdots + \gamma_n\vec{a}_n$

then $\beta_1\vec{a}_1 + \cdots + \beta_k\vec{a}_k = \gamma_1\vec{a}_1 + \cdots + \gamma_k\vec{a}_k \Rightarrow (\beta_1-\gamma_1)\vec{a}_1 + \cdots + (\beta_k-\gamma_k)\vec{a}_k = \vec{0}$

but $\vec{a}_i$'s are l. indep. $\beta_i - \gamma_i = 0 \Rightarrow \beta_i = \gamma_i$

converse also true

Linear Independence

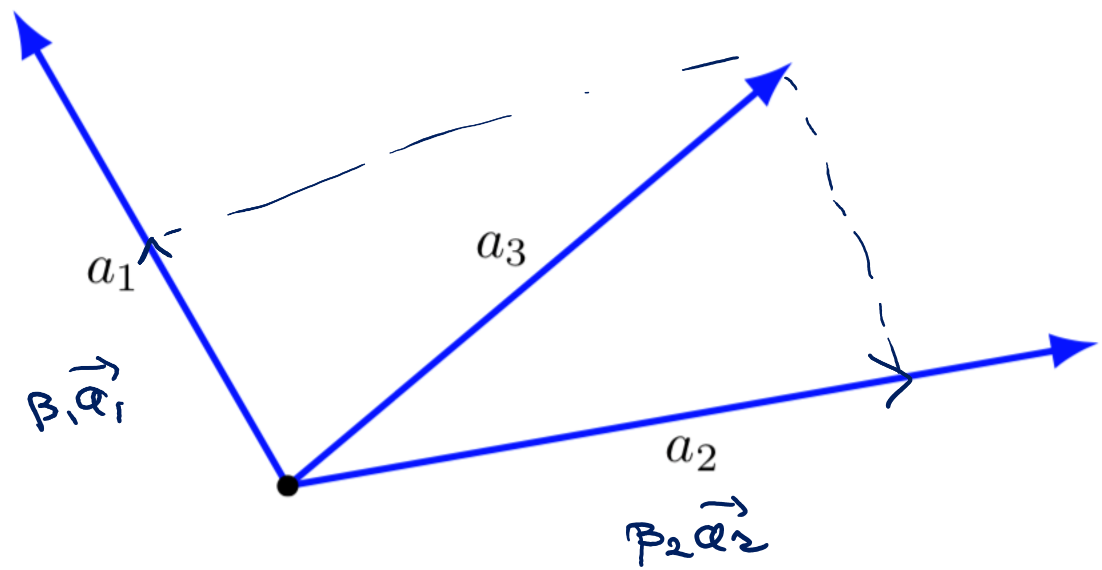
- Supersets: $\{\vec{a_1}, \dots, \vec{a_k}\}$ L.D

  $\{\vec{a_1}, \dots, \vec{a_k}, \vec{\beta_1}, \dots, \vec{\beta_m}\}$ is L.D ~~or L.I~~?

- Subset : $\{\vec{a_1}, \dots, \vec{a_k}\}$ L.I

  take $\{\vec{a_1}, \dots, \vec{a_j}\}$ $j < k$ $\rightarrow$ L.I

Linear Independence



$$\vec{a_3} = \beta_1 \vec{a_1} + \beta_2 \vec{a_2}$$

Vector Space, Subspace

- Vector space X over a field $\mathcal{F}$ (elements are called scalars) is a collection (set) of elements called vectors, equipped with two binary operations
  1. vector addition
  2. scalar mult.

  satisfying a bunch of properties
  1. closure: if $\vec{x}, \vec{y} \in X$, $a \in \mathcal{F}$ then $\vec{x}+\vec{y} \in X$, $a\vec{x} \in X$

  + 7 more properties

eg: $\mathbb{R}^n = \{(x_1, \dots, x_n) : x_j \in \mathbb{R}\}$

$\mathbb{C}^n = \{(x_1, \dots, x_n) : x_j \in \mathbb{C}\}$

Q: what is a subspace of $\mathbb{R}^3$?

$\mathbb{R}^2$ is indeed ↗ $\begin{pmatrix} * \\ * \\ 0 \end{pmatrix}$ or $\begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix}$

$\mathbb{R}$

Subspace of vector space

(i) $S \subset X$

(ii) if $\vec{x}, \vec{y} \in S$ then $a\vec{x} + b\vec{y} \in S$

Span of $\vec{x}_1 \dots \vec{x}_n$

$S = \langle \vec{x}_1, \dots, \vec{x}_k \rangle$ = set of all linear combinations of $\vec{x}_i$

$c_1\vec{x}_1 + \dots + c_k\vec{x}_k \in S$

Linear Span

Basis

Independence-dimension inequality

How many L.I vectors in n-dim?

"A linearly independent collection of n-vectors can have at most n elements"

OR: Any collection of $n+1$ or more n-vectors, is linearly dependent

eg $\mathbb{R}^n$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 10 \\ 0 \\ \vdots \\ 5 \end{bmatrix} = 10\vec{e}_1 + 5\vec{e}_n$$

Basis (def): A collection of n linearly indep. n-vectors (i.e. a collection of L.I vectors of maximum size)

if $\{\vec{a}_1, \ldots, \vec{a}_n\}$ is basis then any vector $\vec{b}$ in our vector space can be written as a linear combination of them

Basis

consider $\{\vec{a_1}, \dots, \vec{a_n}, \vec{b}\}$ n+1 vectors

by independence-dimension ineq $\Rightarrow$ they are L.D., $\underline{\exists\, \beta_1, \dots, \beta_{n+1} \text{ not all zero,}}$

$$\beta_1 \vec{a_1} + \dots + \beta_n \vec{a_n} + \beta_{n+1} \vec{b} = \vec{0}$$

Q: can $\beta_{n+1} = 0$?

if $\beta_{n+1} = 0$, $\beta_1 \vec{a_1} + \dots + \beta_n \vec{a_n} = \vec{0}$

since $\vec{a_i}$ L.I, it implies $\beta_1 = \dots = \beta_n = 0 = \beta_{n+1}$ $\rightarrow\leftarrow$

conclude $\beta_{n+1} \neq 0$

$$\vec{b} = \underbrace{\left(-\beta_1/\beta_{n+1}\right)}_{\gamma_1} \vec{a_1} + \dots + \underbrace{\left(-\beta_n/\beta_{n+1}\right)}_{\gamma_n} \vec{a_n}$$

- expansion of $\vec{b}$ in basis

<u>unique</u>

- γ_i coefficients of the expansion

Basis

eg

① $\{\vec{e_i}\}$, $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = b_1\vec{e_1} + \cdots + b_n\vec{e_n}$

② $\mathbb{R}^2$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ "standard basis"

$\vec{a}_1 = \begin{bmatrix} 1.2 \\ -2.6 \end{bmatrix}, \begin{bmatrix} -0.3 \\ -3.7 \end{bmatrix}$ is a basis

$= \vec{a}_2$

$\vec{e}_1 = \beta_1 \vec{a}_1 + \beta_2 \vec{a}_2$

$\vec{e}_2 = \beta'_1 \vec{a}_1 + \beta'_2 \vec{a}_2$

Basis

Proof of independence-dimension ineq.

by induction on dimension n.

- Base case: $n=1$. Consider a collection of 1-dim vectors $a_1, \dots, a_k$ that are linearly ind. means $a_1 \neq 0$ so for every $\boxed{a_i = \frac{a_i}{a_1} \cdot a_1}$ contradict L.I unless $k=1$.

- I.H: Suppose, for $n \geq 2$, the independence-dim inequality holds for dimension $\leq n-1$.

- I.S: Need to show that it holds for dim n. meaning for any collection of L.I vectors $\{\vec{a}_1, \dots, \vec{a}_k\}$, $k \leq n$.

Linear Independence

$\vec{a}_1, \dots, \vec{a}_k$ L.I

$\vec{a}_i = \begin{bmatrix} \vec{b}_i \\ a_i[n] \end{bmatrix} \quad \forall i$, where $\vec{b}_i = \begin{bmatrix} a_i[1] \\ \vdots \\ a_i[n-1] \end{bmatrix}$ $i = 1, \dots, k$

$\vec{b}_i \to$ n-1 dim

$a_i[n] \uparrow$ scalar

case 1

- First, suppose that $a_1[n] = a_2[n] = \dots = a_k[n] = 0$

Then $\vec{b}_1, \dots, \vec{b}_k$ are linearly indep. b/c

$$\sum_{i=1}^{k} \beta_i \vec{b}_i = 0 \iff \sum_{i=1}^{k} \beta_i \vec{a}_i = 0 \Rightarrow \beta_1 = \dots = \beta_k = 0$$

by induction hypothesis for n-1 dimensions ($\vec{b}_i$ are n-1-dim)

$k \leq n-1$, so definitely $k \leq n$.

case 2

- Now next step suppose $a_i[n]$ not all zero.

try to recreate case 1 by modifying the vectors.