Linear Algebra

CSCI 2820

Lecture 7

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ECES 122

Today

- Finish independence-dimension inequality
- Orthogonality
- More on vector spaces
- Examples

Refresher on linear independence/basis

Show that the following vectors form a basis for R^3

$$\{\overrightarrow{v_1} = (1,1,1), \overrightarrow{v_2} = (1,1,-1), \overrightarrow{v_3} = (1,-1,-1)\}$$

Basis: ② linearly indep?

① n vectors

$$\left.\begin{array}{l}\beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \beta_3 \vec{v}_3 = \vec{0} \\ \text{only holds for } \beta_1 = \beta_2 = \beta_3 = 0\end{array}\right\} \text{def of L.I}$$

$$\vec{0} = \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \beta_3 \vec{v}_3 = (\beta_1, \beta_1, \beta_1) + (\beta_2, \beta_2, -\beta_2) + (\beta_3, -\beta_3, -\beta_3)$$

$$\Rightarrow \vec{0} = (\beta_1 + \beta_2 + \beta_3, \beta_1 + \beta_2 - \beta_3, \beta_1 - \beta_2 - \beta_3)$$

means: $\left\{\begin{array}{l}\beta_1 + \beta_2 + \beta_3 = 0 \\ \beta_1 + \beta_2 - \beta_3 = 0 \\ \beta_1 - \beta_2 - \beta_3 = 0\end{array}\right\}$

$$\left.\begin{array}{l}2(\beta_1 + \beta_2) = 0 \\ \beta_3 = \beta_1 + \beta_2 \\ \beta_1 = \beta_2 + \beta_3\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\beta_1 = -\beta_2 \\ \beta_3 = 0 \\ \beta_1 = \beta_2\end{array}\right\} \Rightarrow \beta_2 = -\beta_2 \Rightarrow \beta_2 = 0$$

$$\Rightarrow \beta_1 = 0$$

Refresher on linear independence/basis

Express the 3-dim standard basis vectors $\vec{e_1}$, $\vec{e_2}$, $\vec{e_3}$ in the basis $\{\vec{v_1} = (1,1,1), \vec{v_2} = (1,1,-1), \vec{v_3} = (1,-1,-1)\}$

$$\vec{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{e_1} = \beta_1 \vec{v_1} + \beta_2 \vec{v_2} + \beta_3 \vec{v_3}$$ (what are the $\beta_1, \beta_2, \beta_3$?)

suggestion: $\beta_1 = 1, \beta_2 = 0, \beta_3 = 0$

$$\vec{e_1} = 1 \cdot \vec{v_1} + 0 \cdot \vec{v_2} + 0 \cdot \vec{v_3} = \vec{v_1} \quad \times$$

$$\vec{e_1} = \tfrac{1}{2}\vec{v_1} + 0 \cdot \vec{v_2} + \tfrac{1}{2}\vec{v_3} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \checkmark$$

$$\vec{e_1} = \tfrac{1}{2}\vec{v_1} + \tfrac{1}{2}\vec{v_3}$$

Refresher on linear independence/basis

Assume we have a collection of k linearly independent 5-vectors $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_k}\}$, and let $\{\vec{b}_1, \overrightarrow{b_2}, \dots, \overrightarrow{b_k}\}$ be such that:

- $\overrightarrow{v_i} = \begin{bmatrix} \vec{b}_i \\ v_i[n] \end{bmatrix}$. Assume all the $v_i[n]$=0 for all i.

$\vec{v}_i = \begin{bmatrix} \vec{b}_i \\ 0 \end{bmatrix}$

- What is the dimension of the $\vec{b}_i$?
- What can you say about the linear independence of the $\vec{b}_i$?

Independence-dimension, condt.

"at most n L.I vectors in n dim"

Induction on dimension. $\boxed{\text{Prove that if we have } \{a_1, \ldots, a_k\} \text{ L.I } n\text{-vectors then } k \leq n}$

- $\underline{\text{Base case}}$: dim = 1 $\{a_1, \ldots, a_k\}$ of scalars linearly indep $\Rightarrow$ do not contain zero, $a_1 \neq 0 \Rightarrow a_i = \left(\frac{a_i}{a_1}\right) \cdot a_1$
  $\Rightarrow k = 1$ ✓
- $\underline{\text{I.H}}$ Assume "red box" holds for dim < n
- $\underline{\text{I. Proof}}$: show it for dim $= n$.
  $\{\vec{a}_1, \ldots, \vec{a}_k\}$ L.I n-vectors, $\vec{a}_i = \begin{bmatrix} \underbrace{\vec{b}_i}_{\to n-1 \text{ dim}} \\ \underbrace{a_i[n]}_{\to \text{scalar}} \end{bmatrix}$
- $\underline{\text{Case 1}}$: all $a_i[n] = 0$. $\Rightarrow \vec{b}_i$'s L.I. since $\vec{b}_i$'s are $n-1$ dim $\Rightarrow$ (by I.H) $k \leq n-1 < n$ ✓
- $\underline{\text{case 2}}$: assume there is some j s.t $a_j[n] \neq 0$

Independence-dimension, condt.

Define following $k-1$ vectors: $a_j[n] \neq 0$

$$\vec{c_i} = \vec{a_i} - \frac{a_i[n]}{a_j[n]}\vec{a_j} \qquad i \neq j$$

$$= \begin{bmatrix} \vec{b_i} - \frac{a_i[n]}{a_j[n]}\vec{b_j} \\ a_i[n] - \frac{a_i[n]\cdot a_j[n]}{a_j[n]} \end{bmatrix} = \begin{bmatrix} \vec{b_i} - \frac{a_i[n]}{a_j[n]}\vec{b_j} \\ 0 \end{bmatrix}.$$

We will show that $\{\vec{c_i}\}$ are L.I.

$$\boxed{\sum_{i\neq j}\beta_i \vec{c_i} = \vec{0}} \Leftrightarrow \sum_{i\neq j}\beta_i\begin{bmatrix}\vec{b_i}-\frac{a_i[n]}{a_j[n]}\vec{b_j}\\ 0\end{bmatrix} = \vec{0} \Rightarrow$$

$$\sum\beta_i\begin{bmatrix}\vec{b_i}\\ 0\end{bmatrix} \underbrace{- \frac{1}{a_j[n]}\sum_i \beta_i a_i[n]}_{\gamma} \begin{bmatrix}\vec{b_j}\\ 0\end{bmatrix} = \vec{0} \Rightarrow$$

verify: last coord is $\sum_{i\neq j}\beta_i a_i[n] - \frac{1}{a_j[n]}\sum \beta_i a_i[n] a_j[n] = 0$

$$\sum\beta_i\begin{bmatrix}\vec{b_i}\\ 0\end{bmatrix} + \gamma\begin{bmatrix}\vec{b_j}\\ 0\end{bmatrix} = \vec{0}$$

claim: This implies $\sum_{i\neq j}\beta_i \underbrace{\begin{bmatrix}\vec{b_i}\\ a_i[n]\end{bmatrix}}_{\vec{a_i}} + \gamma\underbrace{\begin{bmatrix}\vec{b_j}\\ a_j[n]\end{bmatrix}}_{\vec{a_j}} = \vec{0}$

$\Rightarrow \sum \beta_i \vec{a_i} = 0$ can only happen if $\beta_i = 0$

$\overset{I.H.}{\Rightarrow} k-1 \le n-1 \Rightarrow \boxed{k \le n}$

Orthogonality

- Say that $\{\vec{a}_1, \ldots, \vec{a}_k\}$ is orthogonal if $\vec{a}_i \perp \vec{a}_j \ \forall i \neq j$
- It is orthonormal if orthogonal and $\|\vec{a}_i\| = 1 \ \forall i$

Normalized: $\frac{\vec{a}_i}{\|\vec{a}_i\|}$ has norm 1

$\Downarrow$ $\{\vec{a}_1, \ldots, \vec{a}_k\}$ orthonormal if

$$\langle a_i, a_j \rangle = \begin{cases} 1 & i=j \\ 0 & \text{o.w} \end{cases}$$

examples:

① $\{\vec{e}_1, \ldots, \vec{e}_n\}$ orthonormal

② $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v}_2 = 1/\sqrt{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = 1/\sqrt{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\langle \vec{v}_1, \vec{v}_2 \rangle = 0 = \langle \vec{v}_1, \vec{v}_3 \rangle$

$\langle \vec{v}_2, \vec{v}_3 \rangle = \frac{1}{2}(1-1) = 0$

$\langle \vec{v}_1, \vec{v}_1 \rangle = 1$

$\langle \vec{v}_2, \vec{v}_2 \rangle = \frac{1}{2} \cdot (1^2 + 1^2) = 1$

$\langle \vec{v}_3, \vec{v}_3 \rangle = 1/2 (1^2 + (-1)^2) = 1$

Orthogonality

observation: orthonormal vectors are linearly indep.

$$\underbrace{\beta_1 \vec{a}_1 + \cdots + \beta_k \vec{a}_k}_{} = \vec{0}$$

need to show $\beta_1 = \cdots = \beta_k = 0$

take inner product on both sides:

$$\langle \vec{a}_i, \beta_1 \vec{a}_1 + \cdots + \beta_k \vec{a}_k \rangle = \langle \vec{a}_i, \vec{0} \rangle = 0$$

$$\beta_1 \underbrace{\cancel{\langle \vec{a}_i, \vec{a}_1 \rangle}}_{0} + \cdots + \beta_i \underbrace{\langle \vec{a}_i, \vec{a}_i \rangle}_{1} + \cdots + \beta_k \underbrace{\cancel{\langle \vec{a}_i, \vec{a}_k \rangle}}_{0} = 0$$

$\beta_i = 0$

① $\langle \vec{a}_1, \beta_1 \vec{a}_1 + \cdots + \beta_k \vec{a}_k \rangle = 0 \Rightarrow \beta_1 = 0$

② $\langle \vec{a}_2, \beta_1 \vec{a}_1 + \cdots + \beta_k \vec{a}_k \rangle = 0 \Rightarrow \beta_2 = 0$

$\vdots$

$\beta_k = 0$

Orthogonality

Linear combinations of orthonormal vectors

$$\vec{x} = \beta_1 \vec{a}_1 + \dots + \beta_k \vec{a}_k$$

"expansion of $\vec{x}$ on basis $\{\vec{a}_1, \dots, \vec{a}_k\}$

what are the β_i

take inner product w. $\vec{a}_i$:

$$\langle \vec{a}_i, \vec{x} \rangle = \langle \vec{a}_i, \beta_1 \vec{a}_1 + \dots + \beta_k \vec{a}_k \rangle$$

$$= \beta_1 \underbrace{\langle \vec{a}_i, \vec{a}_1 \rangle}_{0} + \dots + \beta_i \underbrace{\langle \vec{a}_i, \vec{a}_i \rangle}_{1} + \dots + \beta_k \underbrace{\langle \vec{a}_i, \vec{a}_k \rangle}_{0}$$

$$\boxed{\langle \vec{a}_i, \vec{x} \rangle = \beta_i}$$

$$\boxed{\vec{x} = \langle \vec{a}_1, \vec{x} \rangle \vec{a}_1 + \dots + \langle \vec{a}_k, \vec{x} \rangle \vec{a}_k} \quad (*)$$

$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \vec{v}_2 = 1/\sqrt{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = 1/\sqrt{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

orthonormal basis

$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

want: $\vec{x} = \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \beta_3 \vec{v}_3$

$\beta_1, \beta_2, \beta_3$?

$\beta_1 = \langle \vec{v}_1, \vec{x} \rangle = 0 \cdot 1 + 0 \cdot 2 - 3 = -3$

$\beta_2 = \langle \vec{v}_2, \vec{x} \rangle = 1/\sqrt{2} + 2/\sqrt{2} = 3/\sqrt{2}$

$\beta_3 = \langle \vec{v}_3, \vec{x} \rangle = 1/\sqrt{2} - 2/\sqrt{2} = -1/\sqrt{2}$

$\vec{x} = -3\vec{v}_1 + 3/\sqrt{2}\,\vec{v}_2 - 1/\sqrt{2}\,\vec{v}_3$

- RMS, std, avg
- Linear functions
- absolutely anything
- notation practice
- nearest neighbor correlation
- angles
- norms
- cauchy-Schwartz

Vector space, contd.

1.1 Definition A *vector space* (over $\mathbb{R}$) consists of a set V along with two operations '+' and '$\cdot$' subject to the conditions that for all vectors $\vec{v}, \vec{w}, \vec{u} \in V$ and all *scalars* $r, s \in \mathbb{R}$:

(1) the set V is closed under vector addition, that is, $\vec{v} + \vec{w} \in V$

(2) vector addition is commutative, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

(3) vector addition is associative, $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$

(4) there is a *zero vector* $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$

(5) each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$

(6) the set V is closed under scalar multiplication, that is, $r \cdot \vec{v} \in V$

(7) scalar multiplication distributes over scalar addition, $(r+s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$

(8) scalar multiplication distributes over vector addition, $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$

(9) ordinary multiplication of scalars associates with scalar multiplication, $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$

(10) multiplication by the scalar 1 is the identity operation, $1 \cdot \vec{v} = \vec{v}$.

Vector space, contd.

- Example:

Vector space, contd.

Vector space, contd.

Vector space, contd.

Vector space, contd.