Linear Algebra

CSCI 2820

Lecture 8

Prof. Alexandra Kolla

Alexandra.Kolla@Colorado.edu
ECES 122

Today

- Problem solving on chapters 3 and 5
- Vector Spaces
- Gram Schmidt

Refresher on std

$$\mathbf{std}(x) = \frac{\|x - (\mathbf{1}^T x/n)\mathbf{1}\|}{\sqrt{n}}.$$

The standard deviation is a measure of how much the entries of a vector differ from their mean value. Another measure of how much the entries of an n-vector $\vec{x} = (x_1, \dots, x_n)$ differ from each other, called the mean square difference, is defined as

$$MSD(\vec{x}) = \frac{1}{n^2} \sum_{i,j} \left(x_{i+\alpha} - (x_j + \alpha) \right)^2$$

$$\tilde{x} = \vec{x} - \frac{\langle 1, x \rangle}{n} \vec{1}$$

Show that $\boxed{MSD(\vec{x}) = 2std(\vec{x})^2}$

$$\boxed{std(\vec{x}) = \frac{\|\tilde{x}\|}{\sqrt{n}}}$$

- $\boxed{2std(\vec{x})^2 = \frac{2\|\tilde{x}\|^2}{n}}$

$$\mu = \frac{\langle 1, x \rangle}{n} = \frac{\sum x_i}{n}$$

$$\tilde{x} = (x_1 - \mu, \dots, x_n - \mu)$$

- $\vec{x} = (x_1, \dots, x_n)$

$\vec{x}' = (x_1 + \alpha, \dots, x_n + \alpha)$

$MSD(\vec{x}) = MSD(\vec{x}')$?

$MSD(\vec{x}) = MSD(\tilde{x})$

Have to show: $\boxed{MSD(\tilde{x}) = \frac{2\|\tilde{x}\|^2}{n}}$

$MSD(\vec{x}) =$

$$MSD(\tilde{x}) = \frac{1}{n^2}\sum_{i,j=1}^{n}(\tilde{x}_i-\tilde{x}_j)^2 = \frac{1}{n^2}\sum_{i,j=1}^{n}(\tilde{x}_i^2+\tilde{x}_j^2-2\tilde{x}_i\tilde{x}_j)$$

$$= \frac{1}{n^2}\Big[\underbrace{n\cdot\sum_{i=1}^{n}\tilde{x}_i^2 + n\cdot\sum_{j=1}^{n}\tilde{x}_j^2}_{2n\sum_{i=1}^{n}\tilde{x}_i^2} - \underbrace{2\sum_{i,j=1}^{n}\tilde{x}_i\tilde{x}_j}\Big]$$

$$\tilde{x} = \left(x_1-\frac{\sum x_i}{n}, \ldots, x_n-\frac{\sum x_i}{n}\right)$$

$$\sum\tilde{x}_i = \sum x_i - n\cdot\frac{\sum x_i}{n} = \sum x_i - \sum x_i = 0$$

$$= \frac{2n}{n^2}\sum_{i=1}^{n}\tilde{x}_i^2 - \cancel{\frac{2}{n^2}\sum_{i=1}^{n}\tilde{x}_i\cdot\sum_{j=1}^{n}\tilde{x}_j} \to 0$$

Q: what is $\boxed{\sum_{j=1}^{n}\tilde{x}_j = 0} = avg(\tilde{x}) = 0$

$$= \frac{2\sum\tilde{x}_i^2}{n} = \frac{2\|\tilde{x}\|^2}{n}$$

Refresher on linear independence/basis

Suppose $\overrightarrow{x_1}, \ldots, \overrightarrow{x_k}$ are orthonormal n-vectors, and $\vec{x} = \beta_1\overrightarrow{x_1} + \cdots + \beta_k\overrightarrow{x_k}$,
Where β1,…,βk are scalars. Express ‖ $\vec{x}$ ‖ in terms of β = (β1, …, βk).

$$\|\vec{x}\|^2 = \langle x, x\rangle = \langle \beta_1\vec{x}_1 + \cdots + \beta_k\vec{x}_k, \beta_1\vec{x}_1 + \cdots + \beta_k\vec{x}_k\rangle$$

$$= \sum_{i,j=1}^{k} \beta_i\beta_j \langle x_i, x_j\rangle = \sum_{i=1}^{n} \beta_i^2 \underbrace{\langle x_i, x_i\rangle}_{1} = \sum_{i=1}^{n} \beta_i^2$$

$$\langle x_i, x_j\rangle = \begin{cases} 0 & i \neq j \quad \text{by statement} \\ \langle x_i, x_i\rangle = \|\vec{x}_i\|^2 = 1 & i = j \end{cases}$$

$$\vec{x} = \beta_1\vec{x}_1 + \cdots + \beta_k\vec{x}_k$$

$$\|\vec{x}\| = \boxed{\sqrt{\sum_{i=1}^{k} \beta_i^2}}$$

$$\|$$

$$\|\vec{x}\| = \boxed{\sqrt{\sum_{i=1}^{n} x_i^2}}$$

$$\boxed{\vec{x} = x_1\vec{e}_1 + \cdots + x_k\vec{e}_k}$$

Vector space

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},\ x_i \in \mathbb{R} \right\}$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} \in \mathbb{R}^3$$

1.1 Definition A *vector space* (over $\mathbb{R}$) consists of a set V along with two operations '$+$' and '$\cdot$' subject to the conditions that for all vectors $\vec{v}, \vec{w}, \vec{u} \in V$ and all *scalars* $r, s \in \mathbb{R}$:

(1) the set V is closed under vector addition, that is, $\vec{v} + \vec{w} \in V$

(2) vector addition is commutative, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

(3) vector addition is associative, $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$

(4) there is a *zero vector* $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$

(5) each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$

(6) the set V is closed under scalar multiplication, that is, $r \cdot \vec{v} \in V$

(7) scalar multiplication distributes over scalar addition, $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$

(8) scalar multiplication distributes over vector addition, $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$

(9) ordinary multiplication of scalars associates with scalar multiplication, $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$

(10) multiplication by the scalar 1 is the identity operation, $1 \cdot \vec{v} = \vec{v}$.

for $\mathbb{R}^n \Rightarrow \vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$r = 3$

$s = 2$

$\vec{v} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$ $(rs) \cdot \vec{v} = 6 \cdot \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \\ 30 \end{pmatrix}$

$r \cdot (s\vec{v}) = 3 \cdot \left(2 \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}\right) = 3 \cdot \begin{pmatrix} -2 \\ 2 \\ 10 \end{pmatrix}$ ✓

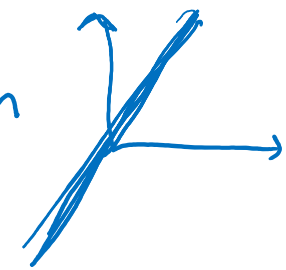
Vector space, contd.

- Example: subset of $\mathbb{R}^2$

$V = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$ <u>not</u> V. Space

(e.g $\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} \notin V$

$L = \left\{ \begin{pmatrix}x\\y\end{pmatrix} \mid y = 3x \right\}$ line through origin



$+, \cdot$ "regular" vector addition, scalar-vector mult of $\mathbb{R}^2$

$\begin{pmatrix}x_1\\y_1\end{pmatrix} + \begin{pmatrix}x_2\\y_2\end{pmatrix} = \begin{pmatrix}x_1+x_2\\y_1+y_2\end{pmatrix}$; $r\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}rx\\ry\end{pmatrix}$

(L "inherits" these operations)

• ex: $K = \left\{ \begin{pmatrix}x\\y\end{pmatrix} \text{ st } x, y \text{ are integers} \right\}$ is this a V. space with $+, \cdot$ over reals?

$0.5 \cdot \begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}0.5\\1\end{pmatrix} \notin K$ NOT. V space!

Vector space, contd.

(1) closure under +

$\vec{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, $\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

$\underline{y_1 = 3x_1}$ $\underline{y_2 = 3x_2}$

need

$y_1 + y_2 = 3(x_1 + x_2) = 3x_1 + 3x_2$ ✓

(2), (3) straightforward from $\mathbb{R}^2$.

(4) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

↑

0-vector $0 = 3\,0$

(5) inverse under + :

$$\underbrace{\begin{pmatrix} -x \\ -y \end{pmatrix}}_{\in L} + \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{if } L} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad ✓$$

if $y = 3x$ then $-y = 3(-x)$

Vector space, contd.

$\left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$ this is the trivial vector space

Vector space, contd.

Vector space, contd.

Gram Schmidt Algo

given: a list of n-vectors $\vec{a_1}, \dots, \vec{a_k}$

for $i = 1 \dots k$

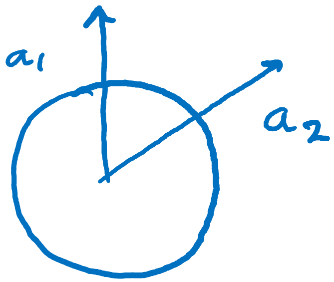
1. Orthogonalization $\tilde{q}_i = \vec{a_i} - \langle q_1, a_i \rangle \vec{q_1} - \dots - \langle q_{i-1}, a_i \rangle \vec{q_{i-1}}$
2. Test for linear dependence. If $\tilde{q}_i = 0$, then quit
3. Normalization, $\vec{q_i} = \tilde{q_i} / \|\tilde{q_i}\|$

For step 1: $i = 1$: $\tilde{q}_1 = \vec{a_1}$, $q_1 = \dfrac{\vec{a_1}}{\|\vec{a_1}\|}$

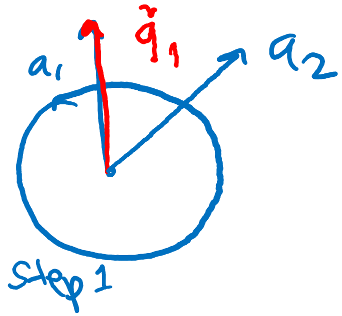
claim1: If algo doesn't quit at any iteration then $\vec{a_i}$ are indeed L.I.

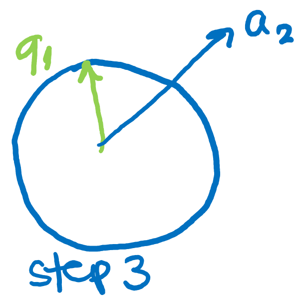
claim2: If algo quits before the end (say $\tilde{q}_j = 0$) Then $\vec{a_i}$ is Linearly dep. and $\vec{a_j} = c_1\vec{a_1} + \dots + c_{j-1}\vec{a_{j-1}}$

$\vec{a}_1, \vec{a}_2 \in \mathbb{R}^2$

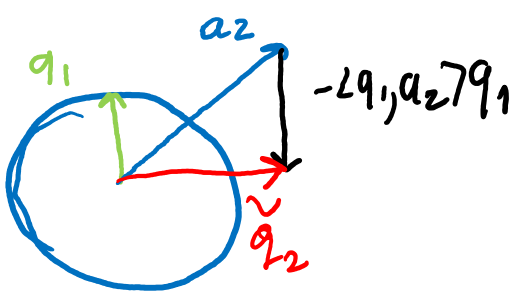


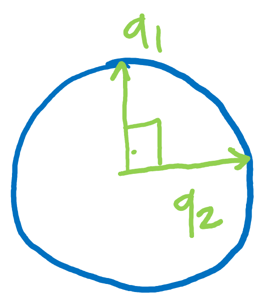
$i=1$





$i=2$





The following hold for Gram-Schmidt

for $i = 1, \dots, k$, assuming $\vec{a}_1, \dots, \vec{a}_k$ are linearly indep.

1) $\tilde{q}_i \neq \vec{0}$ so the linear dependence test in step 2 fails.

2) $\vec{q}_1, \dots, \vec{q}_i$ are orthonormal

3) $\vec{a}_i$ is a linear combination of $\vec{q}_1, \dots, \vec{q}_i$

4) $\vec{q}_i$ is a linear combination of $\vec{a}_1, \dots, \vec{a}_i$