Languages and Regular expressions

Lecture 3

Alphabet, Strings, and Languages

- An alphabet $\Sigma = \{a, b, c\}$ is a finite set of letters/symbols.
- A string over an alphabet Σ is finite sequence of symbols, e.g.
  - sequences $cab, baa,$ and aaa are some strings over $\Sigma = \{a, b, c\}$
  - sequences $\epsilon, 0, 1, 00$, and 01 are some strings over $\Sigma = \{0, 1\}$
- Σ^* is the set of all strings over Σ, e.g. $aabbaa \in \Sigma^*$,
- Naturally, A language L is a collection/set of strings over some alphabet, i.e. $L \subseteq \Sigma^*$ e.g.,
  - $L_{even} = \{w \in \Sigma^* : w$ is of even length$\}$
  - $L_{\{a^n b^n\}} = \{w \in \Sigma^* : w$ is of the form $a^n b^n$ for $n \geq 0\}$

Sets of strings: Σ^n, Σ^*, and Σ^+

- Σ^n is the set of all strings over Σ of length exactly n. Defined inductively as:
  - $\Sigma^0 = \{\varepsilon\}$
  - $\Sigma^n = \Sigma\Sigma^{n-1}$ if $n > 0$
- Σ^* is the set of all finite length strings:

$$\Sigma^* = \bigcup_{n\geq 0} \Sigma^n$$

- Σ^+ is the set of all <u>nonempty</u> finite length strings:

$$\Sigma^+ = \bigcup_{n\geq 1} \Sigma^n$$

Σ^n, Σ^*, and Σ^+

- $|\Sigma^n| = |\Sigma|^n$

- $|\varnothing^n| = ?$
  - $\varnothing^0 = \{\varepsilon\}$
  - $\varnothing^n = \varnothing\varnothing^{n-1} = \varnothing$ if $n > 0$

- $|\varnothing^n| = 1$ if $n = 0$
  $|\varnothing^n| = 0$ if $n > 0$

Σ^n, Σ_*, and Σ^+

- $|\Sigma^*| = ?$
  - Infinity. More precisely, $\aleph_0$
  - $|\Sigma^*| = |\Sigma^+| = |N| = \aleph_0$
- How long is the longest string in Σ^*?

no longest string!

- How many infinitely long strings in Σ^*?

none

Languages

Language

- Definition: A formal language L is a set of strings over some finite alphabet Σ or, equivalently, an arbitrary subset of Σ^*. *Convention: Italic Upper case letters denote languages.*
- Examples of languages :
  - the empty set Ø
  - the set $\{\varepsilon\}$,
  - the set {0,1}* of all boolean finite length strings.
  - the set of all strings in {0,1}* with an odd number of 1's.
  - The set of all python programs that print "Hello World!"
- There are uncountably many languages (but each language has countably many strings)

1	ε	***0***
2	0	***0***
3	1	***1***
4	00	***0***
5	01	***1***
6	10	***1***
7	11	***0***
8	000	***0***
9	001	***1***
10	010	***1***
11	011	***0***
12	100	***1***
13	101	***0***
14	110	***0***
15	111	***1***
16	1000	***1***
17	1001	***0***
18	1010	***0***
19	1011	***1***
20	1100	***0***

Much ado about nothing

- ε is a string containing no symbols. It is not a language.
- $\{\varepsilon\}$ is a language containing one string: the empty string ε. It is not a string.
- $\emptyset$ is the empty language. It contains no strings.

Building Languages

- Languages can be manipulated like any other set.
- Set operations:
  - Union: $L_1 \cup L_2$
  - Intersection, difference, symmetric difference
  - Complement: $\overline{L} = \Sigma^* \setminus L = \{ x \in \Sigma^* \mid x \notin L\}$
  - (Specific to sets of strings) concatenation: $L_1 \cdot L_2 = \{ xy \mid x \in L_1, y \in L_2 \}$

Concatenation

- $L_1 \cdot L_2 = L_1L_2 = \{ xy \mid x \in L_1, y \in L_2 \}$ (we omit the bullet often)

  e.g. L_1 = { fido, rover, spot }, L_2 = { fluffy, tabby }

  then L_1L_2 ={ fidofluffy, fidotabby, roverfluffy, ...}

$|L_1L_2|$ =?
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L_1 = {a,aa}, $L_2 = \emptyset$
$L_1L_2 = \emptyset$

L_1 = {a,aa}, $L_2 = \{\varepsilon\}$
$L_1L_2 = L_1$

Building Languages

- L^n inductively defined: $L^0 = \{\varepsilon\}, L^n = LL^{n-1}$

Kleene Closure (star) L*

Definition 1: $L^* = \cup_{n\geq 0} L^n$, *the set of all strings obtained by concatenating a sequence of zero or more stings from L*

Building Languages

- L^n inductively defined: $L^0 = \{\varepsilon\}, L^n = LL^{n\text{-}1}$

Kleene Closure (star) L*

Recursive Definition: L* is the set of strings w
such that either
—w= ε or
*— w=xy   for x in L and y in L**

Building Languages

- $\{\varepsilon\}^* = \ ?\quad \emptyset^* = ?$

$\{\varepsilon\}^* = \emptyset^* = \{\varepsilon\}$

- For any other L, the Kleene closure is infinite and contains arbitrarily long strings. It is the smaller superset of L that is closed under concatenation and contains the empty string.
- ***Kleene Plus***

$L^+ = LL^*$, set of all strings obtained by concatenating a sequence of at least one string from L.

– When is it equal to L^* ?

Regular Languages

Regular Languages

- The set of regular languages over some alphabet Σ is defined inductively by:
- L is empty
- L contains a single string (could be the empty string)
- If L_1, L_2 are regular, then $L = L_1 \cup L_2$ is regular
- If L_1, L_2 are regular, then $L = L_1 L_2$ is regular
- If L is regular, then L^* is regular

Regular Languages Examples

- L = any finite set of strings. E.g., L = set of all strings of length at most 10
- $L =$ the set of all strings of 0's including the empty string

- Intuitively L is regular if it can be constructed from individual strings using any combination of union, concatenation and unbounded repetition.

Regular Languages Examples

- Infinite sets, but of strings with “regular” patterns
  - Σ^* (recall: L^* is regular if L is)
  - $\Sigma^+ = \Sigma\Sigma^*$
  - All binary integers, starting with 1
    - $L = \{1\}\{0,1\}^*$
  - All binary integers which are multiples of 37
    - *later*

Regular Expressions

Regular Expressions

- A compact notation to describe regular languages
- Omit braces around one-string sets, use + to denote union and juxtapose subexpressions to represent concatenation (without the dot, like we have been doing).
- Useful in
  - text search (editors, Unix/grep)
  - compilers: lexical analysis

Regular Expressions

- In arithmetic, we can use operations $\times, +$ to build up expressions such as $(5+3)\times 4$
- Similarly, we can use regular operations to build up expressions describing languages, which are called regular expressions.
- E.g $(0 \cup 1)0^*$
- Value of arithmetic expression above is 32.
- Value of a regular expression is a language (which one?)

Inductive Definition

A regular expression r over alphabet Σ is one of the following (L(r) is the language it represents):

Atomic expressions (Base cases)	
$\emptyset$	$L(\emptyset) = \emptyset$
w for w $\in \Sigma^*$	$L(w) = \{w\}$

Inductively defined expressions	
(r_1+r_2)	$L(r_1+r_2) = L(r_1) \cup L(r_2)$
$(r_1 r_2)$	$L(r_1 r_2) = L(r_1)L(r_2)$
(r^*)	$L(r^*) = L(r)^*$

alt notation $(r_1|r_2)$ or $(r_1 \cup r_2)$

Any regular language has a regular expression and vice versa

Regular Expressions

- Can omit many parentheses
  - By following precedence rules :
    star ($*$) before *concatenation* ($\cdot$), before union (+)
    (similar to arithmetic expressions)
    - e.g. $r^*s + t \equiv ((r^*)\, s) + t$
    - 10^* is shorthand for $\{1\}\cdot\{0\}^*$ and NOT $\{10\}^*$
  - By associativity: $(r+s)+t \equiv r+s+t$, $(rs)t \equiv rst$
- More short-hand notation
  - e.g., $r^+ \equiv rr^*$ (note: $^+$ is in superscript)

Regular Expressions: Examples

- $(0+1)^*$
  - All binary strings
- $((0+1)(0+1))^*$
  - All binary strings of even length
- $(0+1)^*001(0+1)^*$
  - All binary strings containing the substring 001
- $0^* \; + \; (0^*10^*10^*10^*)^*$
  - All binary strings with #1s ≡ 0 mod 3
- $(01+1)^*(0+\varepsilon)$
  - All binary strings without two consecutive 0s

Exercise: create regular expressions

- All binary strings with either the pattern 001 or the pattern 100 occurring somewhere

  one answer: (0+1)*001(0+1)* + (0+1)*100(0+1)*

- All binary strings with an even number of 1s

  one answer: 0*(10*10*)*

Regular Expression Identities

- $r^*r^* = r^*$
- $(r^*)^* = r^*$
- $rr^* = r^*r$
- $(rs)^*r = r(sr)^*$
- $(r+s)^* = (r^*s^*)^* = (r^*+ s^*)^* = (r+s^*)^* = \ldots$

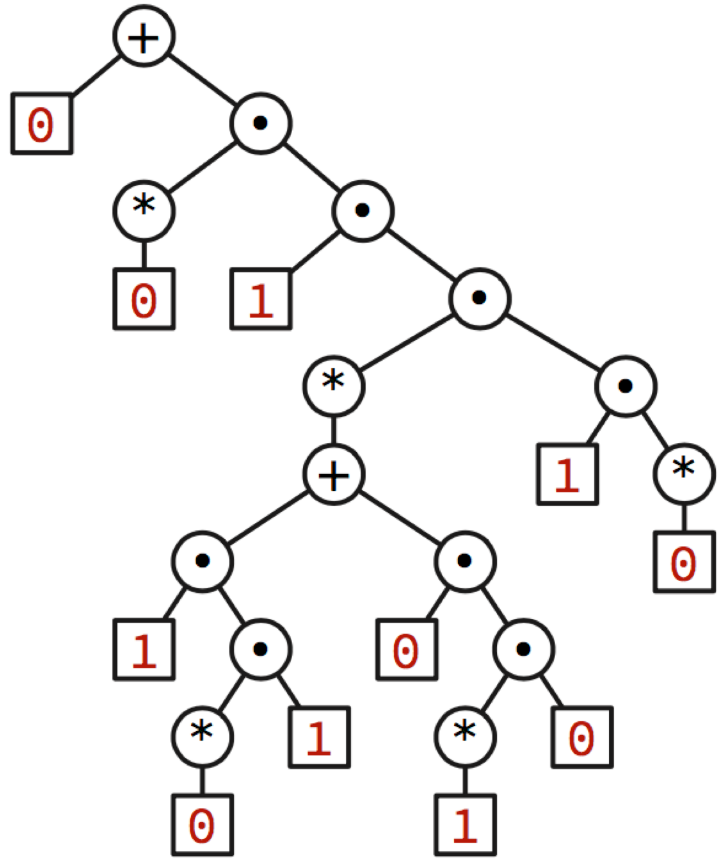
Equivalence

- Two regular expressions are equivalent if they describe the same language. eg.
    - $(0+1)^* = (1+0)^*$ (why?)
- Almost every regular language can be represented by infinitely many distinct but equivalent regular expressions
    - $(L\,\emptyset)^* L\varepsilon + \emptyset = ?$

Regular Expression Trees

- Useful to think of a regular expression as a tree. Nice visualization of the recursive nature of regular expressions.

- Formally, a regular expression tree is one of the following:
  - a leaf node labeled Ø
  - a leaf node labeled with a string
  - a node labeled + with two children, each of which is the root of a regular expression tree
  - a node labeled · with two children, each of which is the root of a regular expression tree
  - a node labeled * with one child, which is the root of a regular expression tree



A regular expression tree for $0 + 0^*1(10^*1 + 01^*0)^*10^*$

Not all languages are regular!

Are there Non-Regular Languages?

- Every regular expression over {0,1} is itself a string over the 8-symbol alphabet {0,1,+,*,(,),ε, Ø}.
- Interpret those symbols as digits 1 through 8. Every regular expression is a base-9 representation of a unique integer.
- Countably infinite!
- We saw (first few slides) there are uncountably many languages over {0,1}.
- In fact, the set of all regular expressions over the {0,1} alphabet is a non-regular language over the alphabet {0,1,+,*,(,),ε, Ø}!!