Context-Free Grammars (and Languages)

Lecture 8

Today

Beyond regular expressions:
Context-Free Grammars (CFGs)

What is a CFG?
What is the language associated with a CFG?

Creating CFGs. Reasoning about CFGs.

Today

First used to study human languages

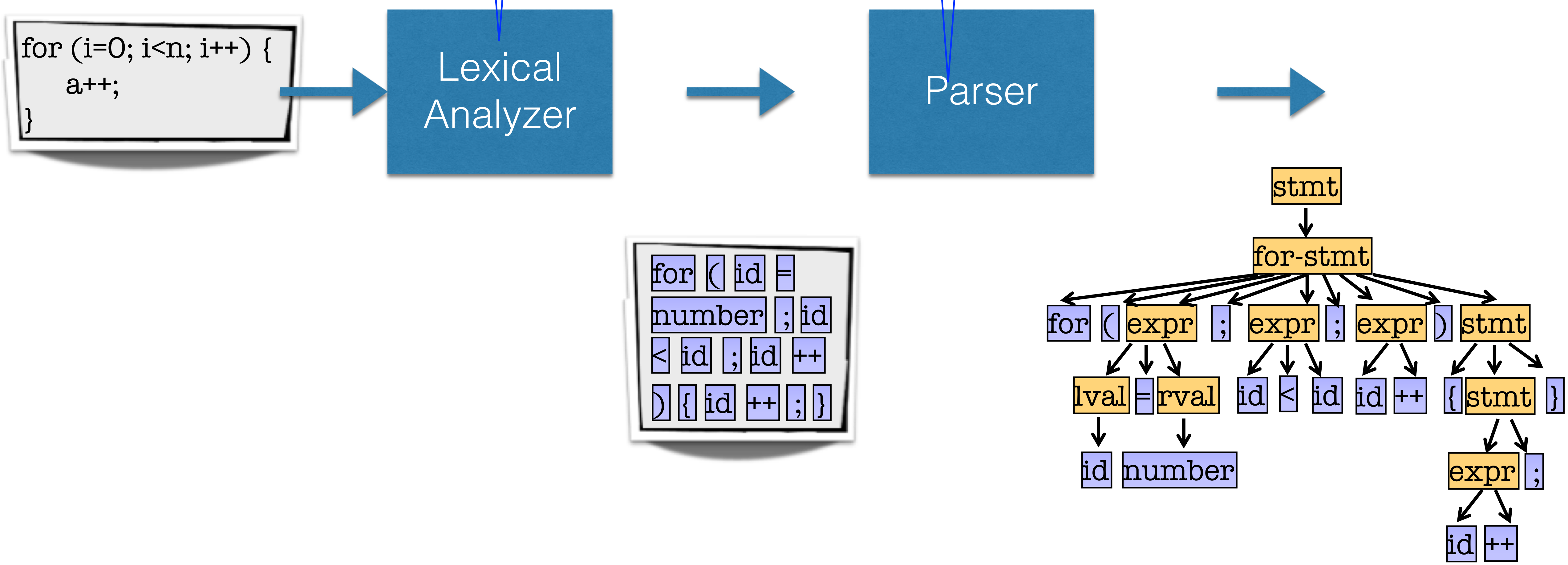
Important applications in specifying and compiling programming languages

Include Regular languages, but much more.

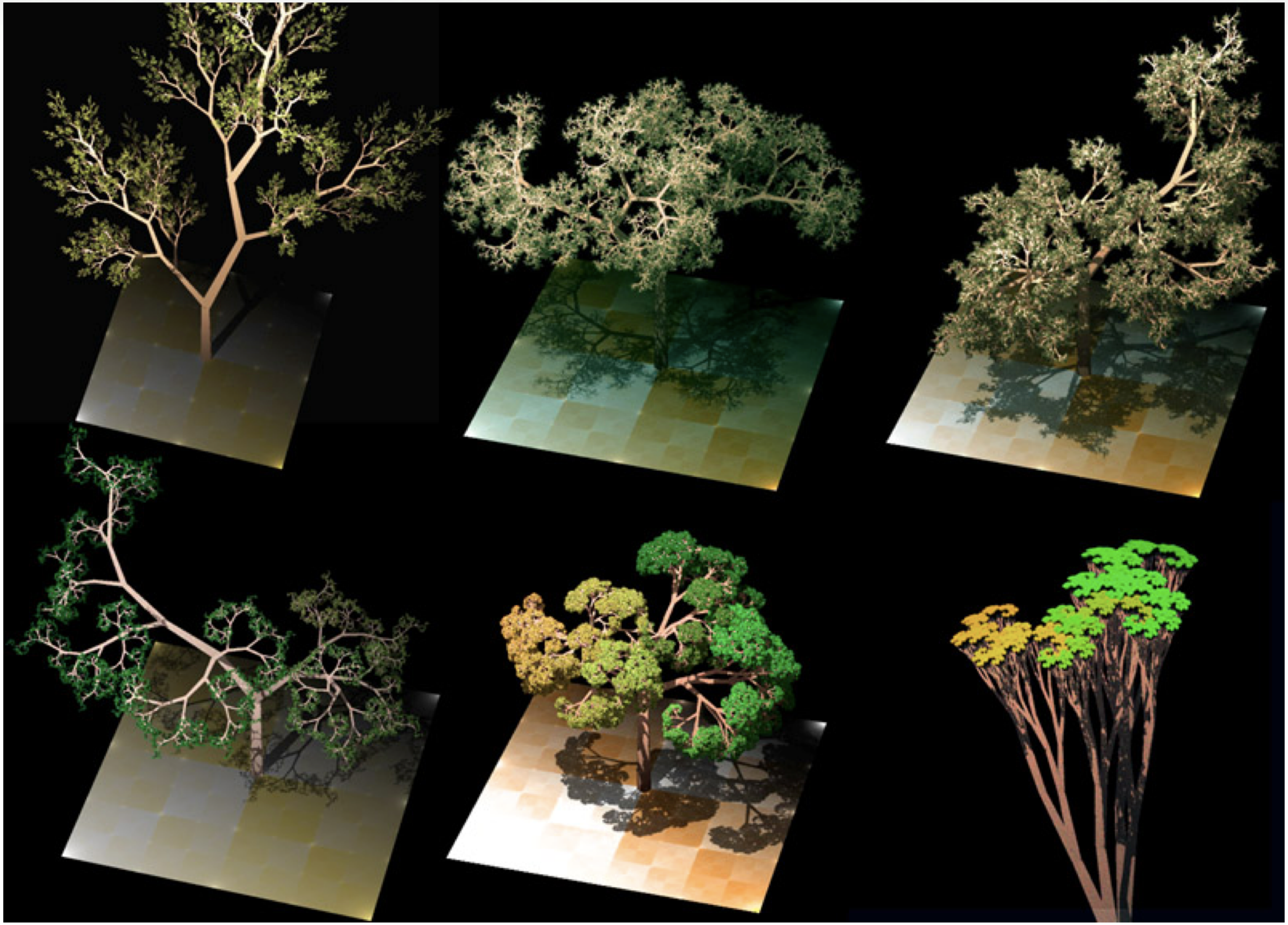
Compiler Frontend

Rules encoded as regular expressions

Rules *cannot be* encoded as regular expressions

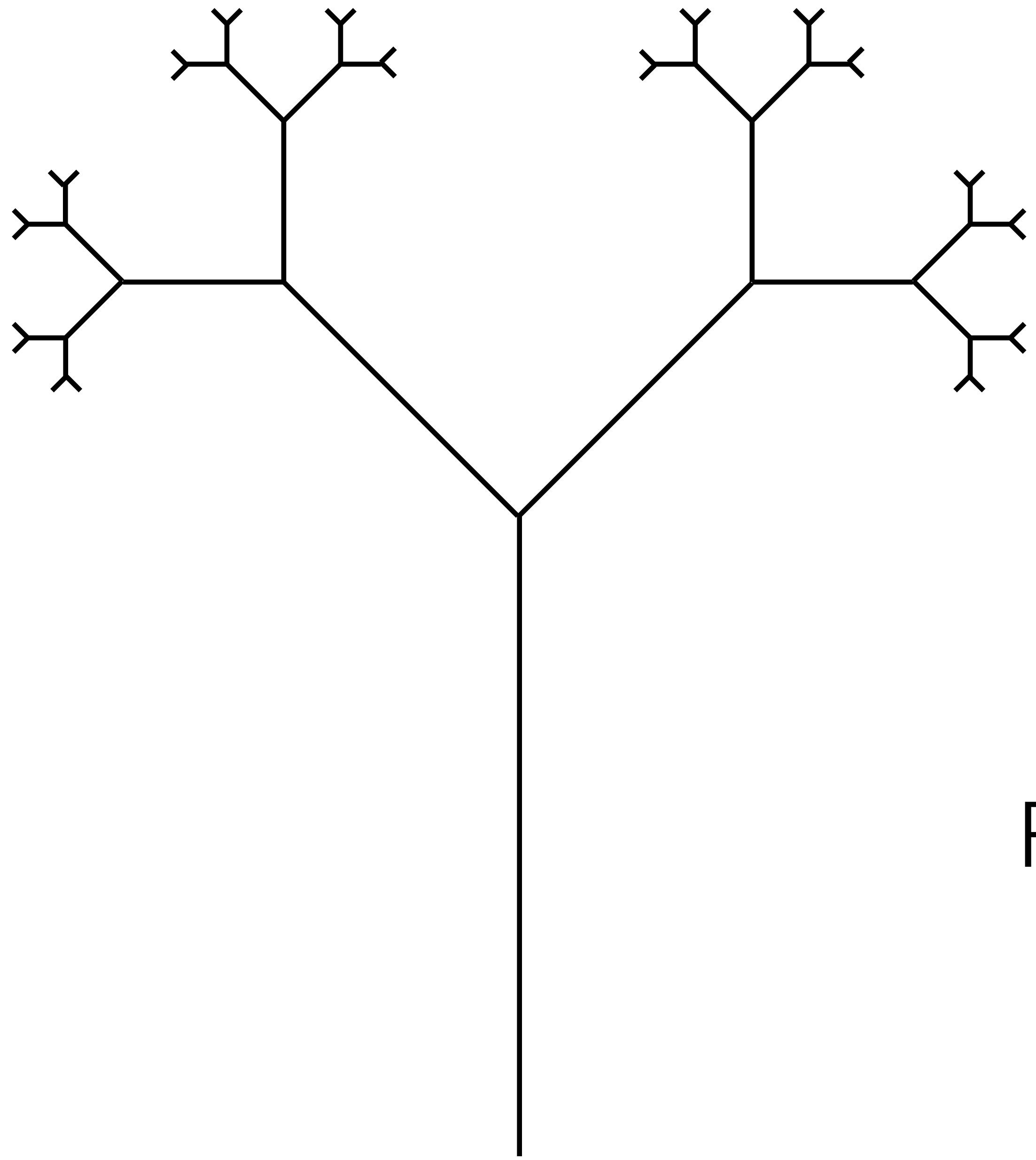


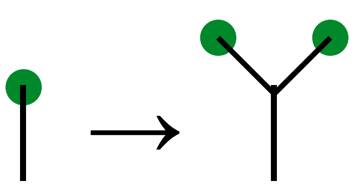
Biological Models



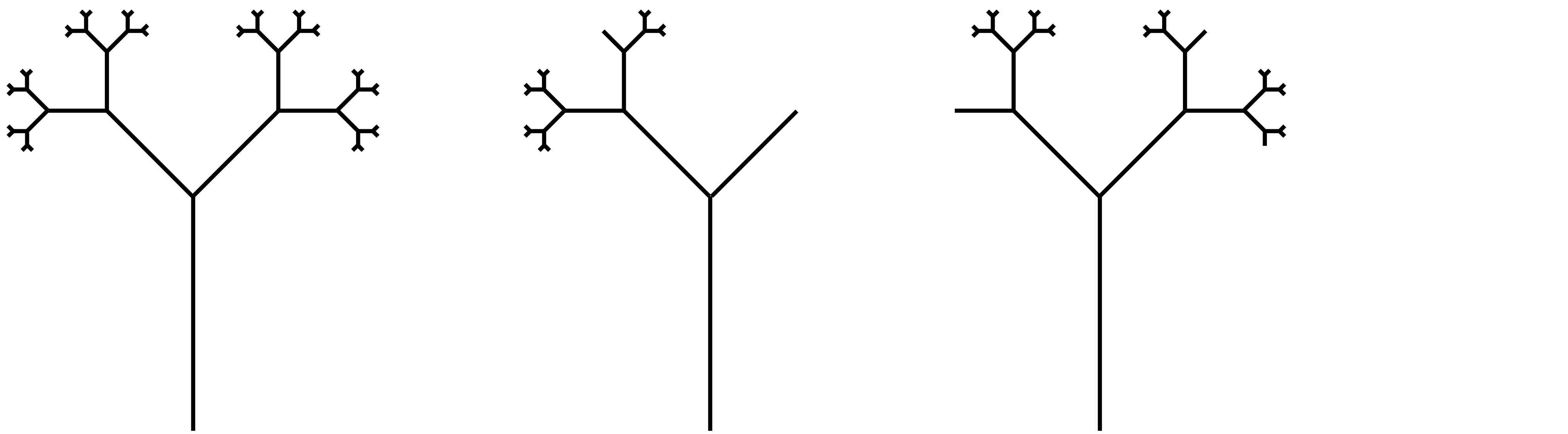
en.wikipedia.org/wiki/**L-system**

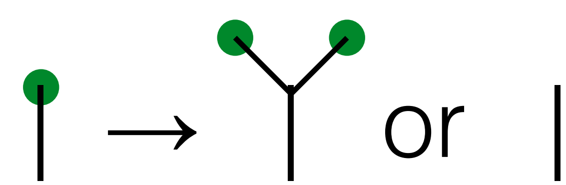
Biological Models



Rule: 

Biological Models



Rule: 

Grammar: ***Rewriting rules*** for generating a set of strings (i.e., a language) from a "seed"

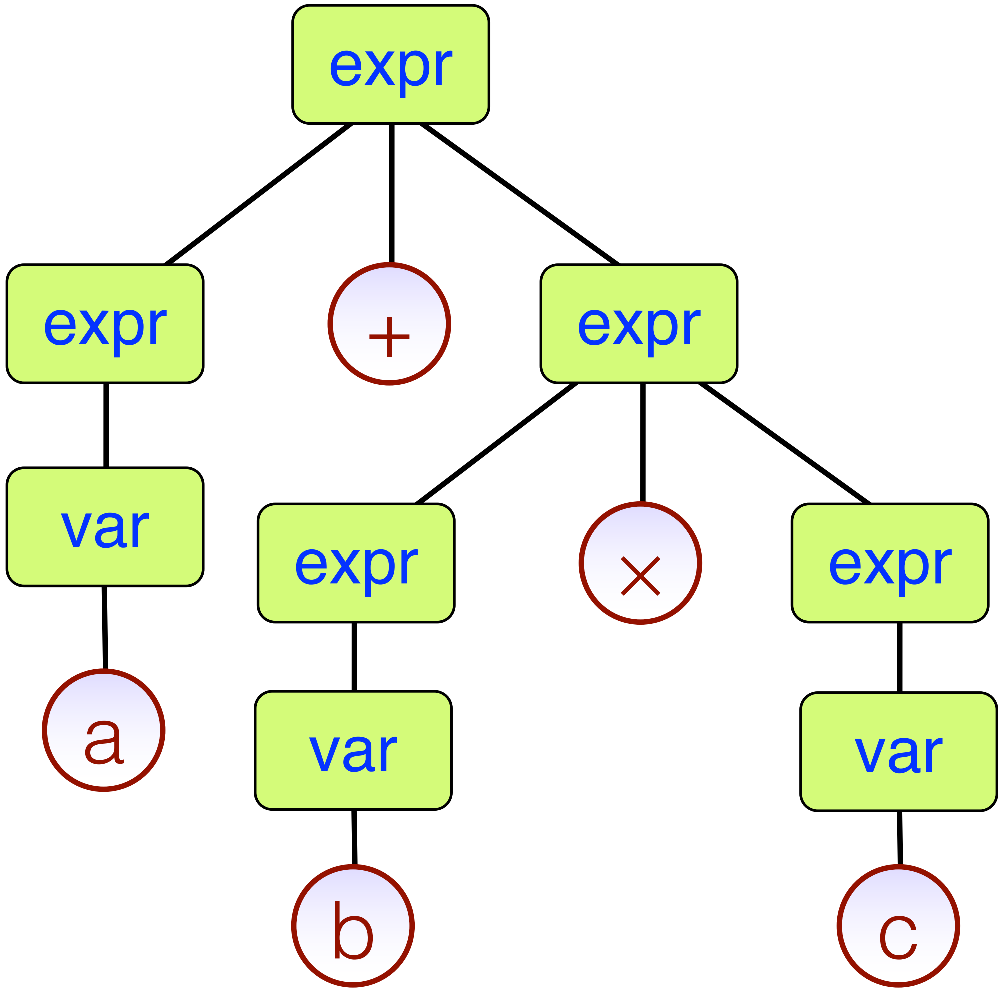
Context-Free Grammar

Example: a (simplistic) syntax for arithmetic expressions

expr → expr + expr
expr → expr × expr
expr → var
var → a
var → b
var → c

e.g. expr ⇒* a + b × c

"derives"



(This grammar is "ambiguous" since there is another parse tree for the same string)

Context-Free Grammar

Example: a (simplistic) syntax for arithmetic expressions

expr → expr + expr
expr → expr × expr
expr → var
var → a
var → b
var → c

expr → expr + expr | expr × expr | var
var → a | b | c

short-hand

e.g. expr $\Rightarrow^*$ a + b × c

"derives"

$$G = (\Sigma, V, P, S)$$

Σ = {a,b,c,+,×} (terminals)

V = {expr, var} (non-terminals)

P = {(A,α) | A → α} (prod. rules)

S = expr (start symbol)

Context-Free Grammar : Arrows

Production Rule: $A \to \pi$, $A \in V$, $\pi \in (\Sigma \cup V)^*$

expr → expr + expr | expr × expr | var
var → a | b | c

Immediately Derives: $\alpha_1 \Rightarrow \alpha_2$ if $\alpha_1, \alpha_2 \in (\Sigma \cup V)^*$
s.t., $\alpha_1 = \beta A \gamma$, $\alpha_2 = \beta \pi \gamma$ and $A \to \pi$

More clearly, if grammar is G, we write $\alpha \Rightarrow_G^* \alpha'$

expr ⇒ expr + expr
expr + expr ⇒ expr + expr × expr

Derives: $\alpha \Rightarrow^* \alpha'$ if $\exists \alpha_1, \ldots, \alpha_{t+1} \in (\Sigma \cup V)^*$ s.t.
$\alpha_1 = \alpha$, $\alpha_{t+1} = \alpha'$, and for all $i \in [1, t]$, $\alpha_i \Rightarrow \alpha_{i+1}$

t-step derivation
$\alpha \Rightarrow^t \alpha'$

Context-Free Grammar : Arrows

Production Rule: $A \rightarrow \pi$, $A \in V$, $\pi \in (\Sigma \cup V)^*$

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expr ⇒ expr + expr
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t-step derivation $\alpha \Rightarrow^t \alpha'$

expr ⇒* expr + expr × expr ⇒* var + var × var ⇒* a + b × c
expr ⇒* a + b × c

Context-Free Languages

The language ***generated*** by a grammar G
with start symbol S and alphabet Σ,

$$L(G) = \{ w \in \Sigma^* \mid S \Rightarrow_G^* w \}$$

Languages generated by a context free grammars are called **Context Free Languages** (CFL)

Examples

Over $\Sigma = \{ 0,1 \}$, give a grammar for the following languages:

- $L = \{ 0^n 1^n \mid n \geq 0 \}$
- $L = \{ w \mid w = w^R \}$
- $L = \{ 0^m 1^n \mid m < n \}$
- $L = \{ 0^m 1^n \mid m \neq n \}$

Examples

Over $\Sigma = \{ 0,1 \}$, give a grammar for the following languages:

- $L = \{ 0^n1^n \mid n \geq 0 \}$

$S \rightarrow \varepsilon \mid 0S1$

- $L = \{ w \mid w = w^R \}$

$S \rightarrow \varepsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1$

- $L = \{ 0^m1^n \mid m < n \}$

$Z \rightarrow \varepsilon \mid 0Z1$ // 0^n1^n
$S \rightarrow Z1 \mid S1$ // 0^m1^n with $m < n$

- $L = \{ 0^m1^n \mid m \neq n \}$

$S \rightarrow A \mid B$
$Z \rightarrow \varepsilon \mid 0Z1$ // 0^n1^n
$A \rightarrow 0Z \mid 0A$ // 0^m1^n with $m > n$
$B \rightarrow Z1 \mid B1$ // 0^m1^n with $m < n$

Parse Tree

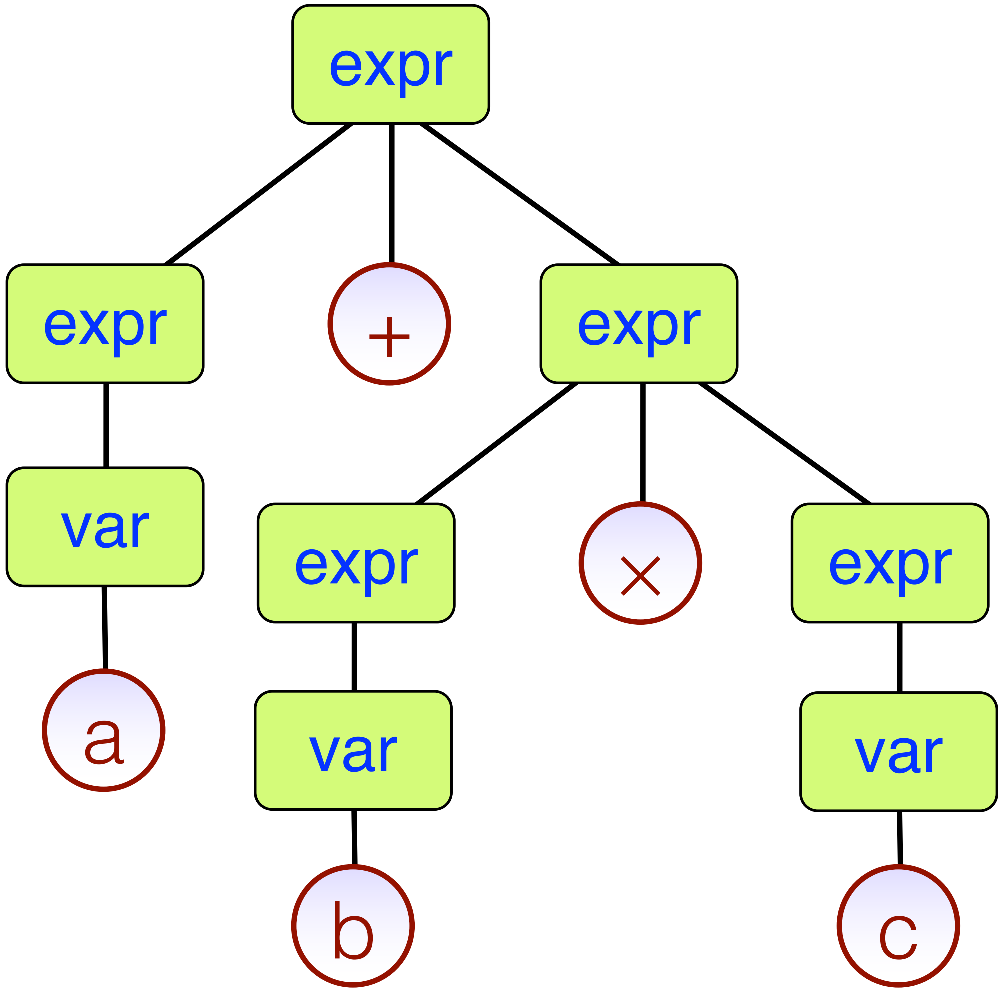
Parse Tree captures the structure of derivations for a given string (but not the exact order)

The exact order of derivations is *not* important

But structure is important!

Ambiguous grammar: If some string has two different parse trees

expr ⇒* a + b × c



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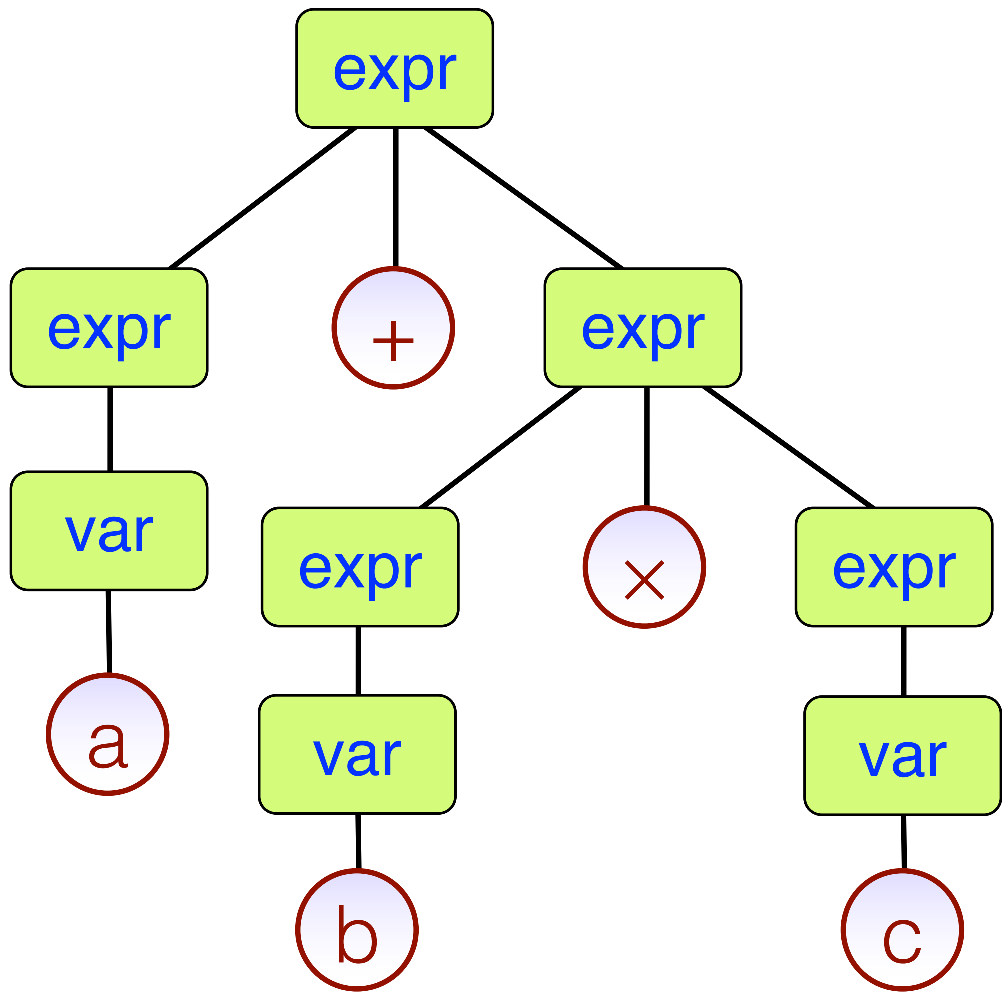
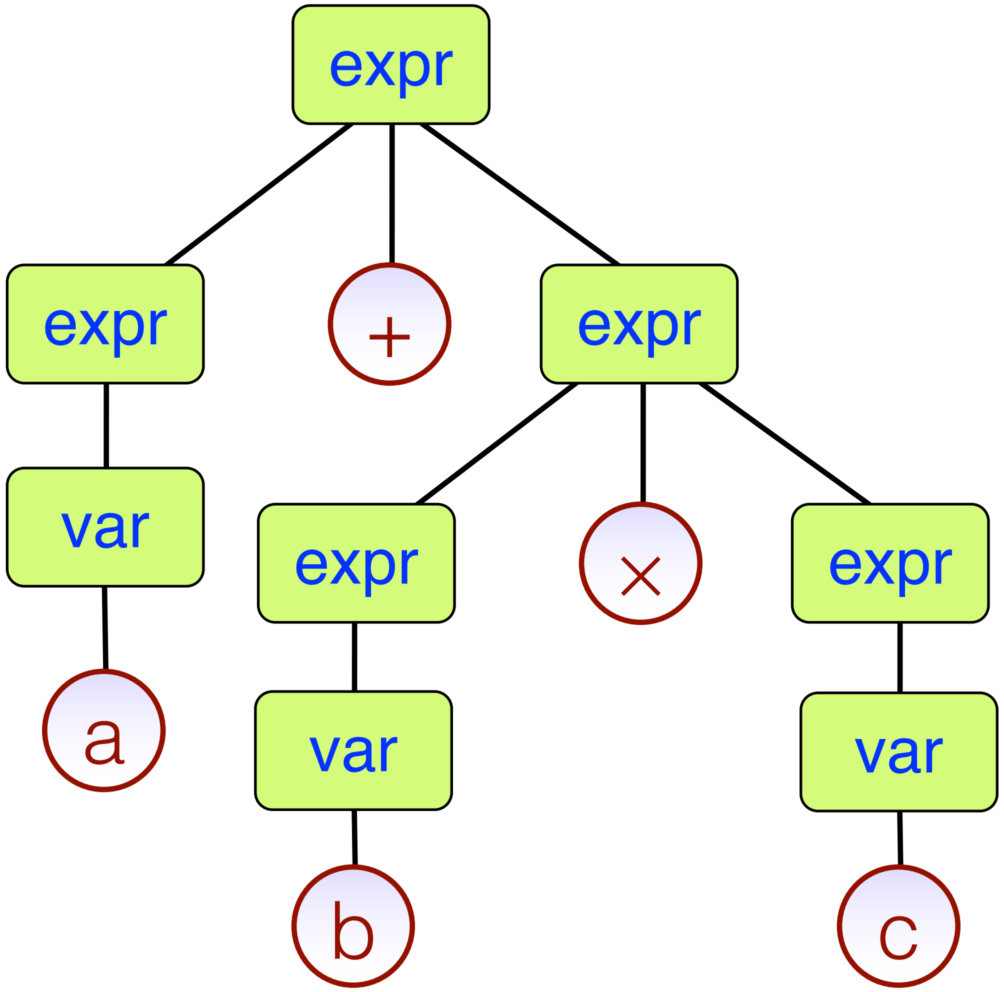


expr ⇒* expr + expr × expr ⇒* var + var × var ⇒* a + b × c
expr ⇒* a + expr ⇒* a + expr × c ⇒* a + b × c

Ambiguity

expr → expr + expr | expr × expr | var
var → a | b | c

expr ⇒* a + b × c



Ambiguity

expr → expr + expr | expr × expr | var
var → a | b | c

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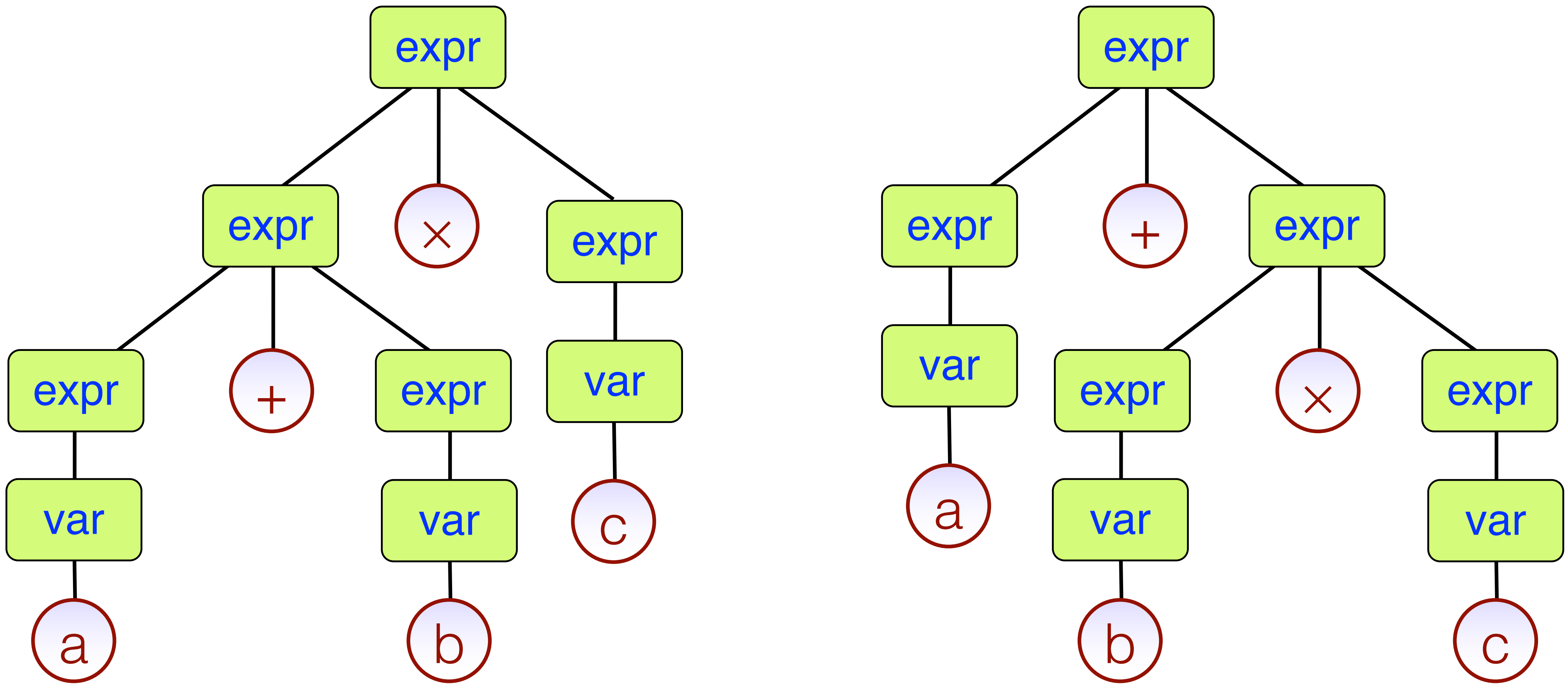


An Unambiguous Grammar

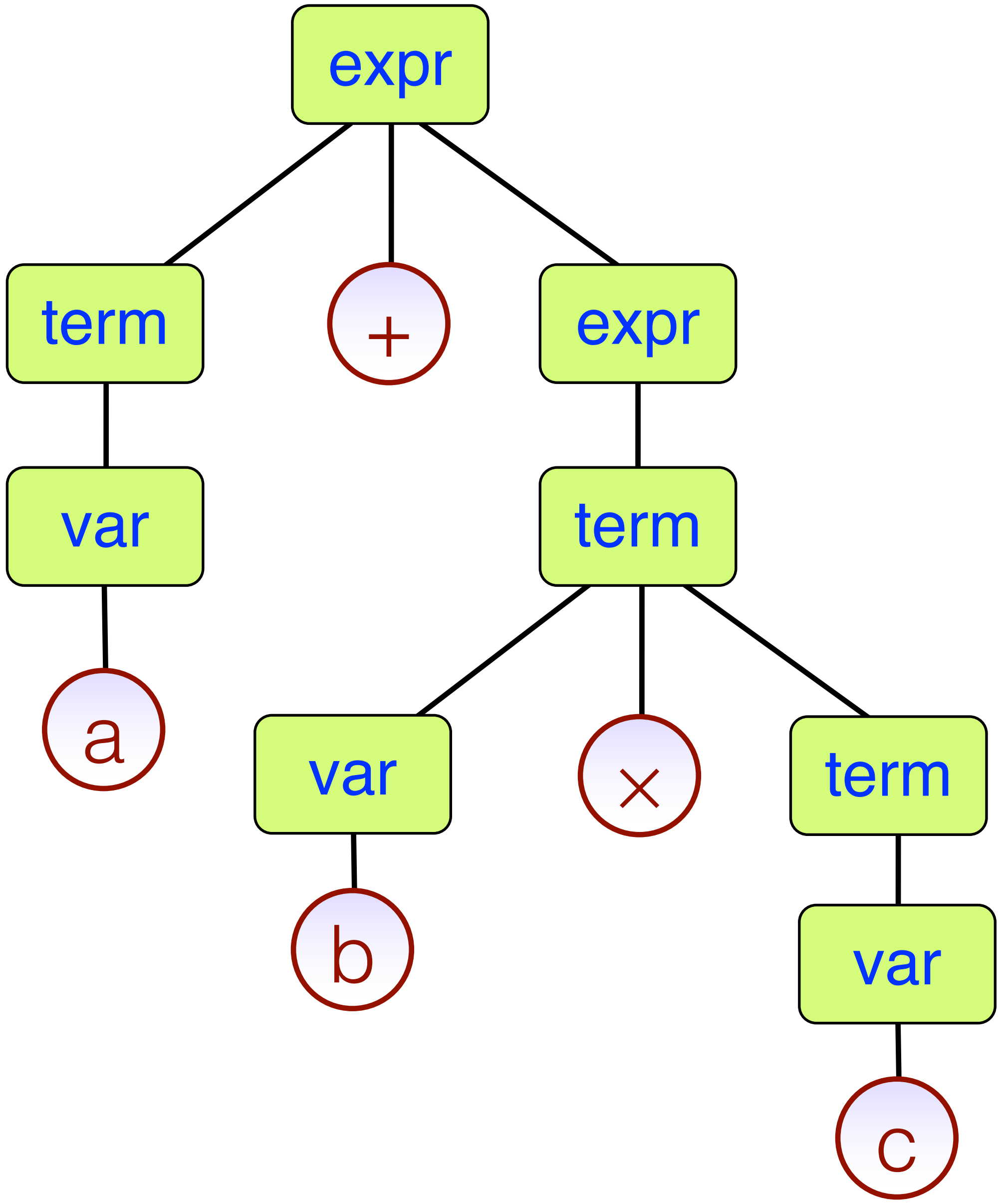
$expr \rightarrow term + expr \mid term$
$term \rightarrow var \mid var \times term$
$var \rightarrow a \mid b \mid c$

$expr \Rightarrow^* a + b \times c$

An Unambiguous Grammar

expr → term + expr | term
term → var | var × term
var → a | b | c

expr ⇒* a + b × c



In practice, unambiguous grammars are important (e.g., in compilers)

Operator precedence enforced by requiring all × carried out (to get a "term") before any +

There are CFLs which do not have *any* unambiguous grammar: inherently ambiguous languages

Examples

- $L = L(0^*)$

$S \to \varepsilon \mid 0 \mid SS$: Ambiguous!

$S \to \varepsilon \mid 0S$: Unambiguous

- L = set of all strings with balanced parentheses

$S \to \varepsilon \mid (S) \mid SS$: Ambiguous!

$T \to (\,) \mid (S)$
$S \to \varepsilon \mid TS$: Unambiguous

Examples

- L = set of all valid regular expressions over $\{0, 1\}$

An ambiguous grammar (start symbol S, $\Sigma = \{\text{Ø}, e, 0, 1, +, *, (,)\}$):
S → Ø | e | 0 | 1 | (S) | S* | SS | S+S

An unambiguous grammar for a *subset* of regular expressions:
S → Ø | e | 0 | 1 | (S) | (S*) | (SS) | (S+S)

Exercise: An unambiguous grammar for *all* valid regular expressions

Proving Correctness of Grammars

Claim: Let $L = \{ w \mid \#_0(w) = \#_1(w) \}$. Then, $L(G) = L$ where the productions of G are: S $\rightarrow 0S1 \mid 1S0 \mid SS \mid \varepsilon$

Challenge: Give an unambiguous grammar

Proof: Need to prove both $L(G) \subseteq L$ and $L(G) \supseteq L$.

Prove $L(G) \subseteq L$ by induction on the length of derivations (or height of parse trees)

Prove $L(G) \supseteq L$ by induction on the length of strings.

Proving Correctness of Grammars

Claim: Let $L = \{ w \mid \#_0(w) = \#_1(w) \}$. Then, $L(G) = L$ where the productions of G are: S → $0S1 \mid 1S0 \mid SS \mid \varepsilon$

Proof: Proving $L(G) \subseteq L$ by induction on the length of derivations.

Let $w \in L(G)$. $S \Rightarrow^t w$ for some $t \geq 1$. Induction on t to show that $w \in L$.

Proving Correctness of Grammars

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Proof: Proving $L(G) \subseteq L$ by induction on the length of derivations.

Let $w \in L(G)$. $S \Rightarrow^t w$ for some $t \geq 1$. Induction on t to show that $w \in L$.
<u>Base case:</u> t=1. Only string derived is ε. ✓

<u>Induction step</u>: Consider $t > 1$. Suppose all u s.t. $S \Rightarrow^k u$, $k < t$, in L.
Let w be such that $S \Rightarrow^t w$. i.e., $S \Rightarrow \alpha_1 \Rightarrow^{t\text{-}1} w$.
<u>Case α_1=0S1</u>: $w = 0u1$ and $S \Rightarrow^{t\text{-}1} u$. By IH, $\#_0(u)=\#_1(u)$.
Hence $\#_0(w) = \#_0(u)+1 = \#_1(v)+1 = \#_1(w)$. (<u>Case α_1=1S0</u> is symmetric.)
<u>Case α_1=SS:</u> $w = uv$ and $S \Rightarrow^m u$, $S \Rightarrow^n v$, $1 \leq m,n < t$ $(m+n = t\text{-}1)$. By IH, $\#_0(u)=\#_1(u)$ & $\#_0(v)=\#_1(v)$. Hence $\#_0(w) = \#_0(u)+\#_0(v) = \#_1(u)+\#_1(v) = \#_1(w)$

Proving Correctness of Grammars

Claim: Let $L = \{ w \mid \#_0(w) = \#_1(w) \}$. Then, $L(G) = L$ where the productions of G are: $S \rightarrow 0S1 \mid 1S0 \mid SS \mid \varepsilon$

Proof: Proving $L(G) \supseteq L$ by induction on the length of strings.

Suppose $w \in L$. To show by induction on $|w|$ that $w \in L(G)$.

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Proving Correctness of Grammars

Claim: Let $L = \{ w \mid \#_0(w) = \#_1(w) \}$. Then, $L(G) = L$ where the productions of G are: S → 0S1 | 1S0 | SS | ε

Proof: Proving $L(G) \supseteq L$ by induction on the length of strings.

Suppose $w \in L$. To show by induction on $|w|$ that $w \in L(G)$.
Base cases: $|w|=0$. $\varepsilon \in L(G)$. ✓ No string with $|w|=1$ in $L(G)$. ✓

Induction step: Let $n \geq 2$. Suppose $u \in L(G)$ for all $u \in L$ with $|u| < n$.
Let $w \in L$ be such that $|w|=n$; i.e., $\#_0(w)=\#_1(w)$.

Case $w=0u1$: Then $u \in L$ and $|u| < n$. By IH, $u \in L(G)$. i.e., $S \Rightarrow^* u$.
Hence, $S \Rightarrow 0S1 \Rightarrow^* 0u1 = w$. (Case $w=1u0$ is symmetric.)
Case $w=0u0$: Let $d_i = \#_0(i\text{-long prefix of } w) - \#_1(i\text{-long prefix of } w)$.
Then $d_1 = 1$, $d_n = 0$, $d_{n-1} = -1$. So $\exists\ 1 < m \leq n-1$ s.t., $d_m = 0$. i.e., $w=xy$, where $|x|, |y| < |w|$, and $x,y \in L$. By IH, $x,y \in L(G)$. Hence $S \Rightarrow SS \Rightarrow^* xy = w$.
(Case $w=1u1$ is symmetric.)

Proving Correctness of Grammars

Often will need to strengthen the claim to include strings generated by every variable in the grammar

Claim: Let $L = \{ w \mid \#_0(w) = \#_1(w) \}$. Then, $L(G) = L$ where productions of G are:

$S \rightarrow AB \mid BA \mid \varepsilon$
$A \rightarrow 0 \mid AS \mid SA$
$B \rightarrow 1 \mid BS \mid SB$

Stronger Claim:
A derives all strings w s.t. $\#_0(w) = \#_1(w)+1$.
B derives all strings w s.t. $\#_1(w) = \#_0(w)+1$.
S derives all strings w s.t. $\#_0(w) = \#_1(w)$.

Closure Properties for CFL

Union: If L_1 and L_2 are CFLs, so is $L_1 \cup L_2$.

Let $G_1 = (\Sigma, V_1, P_1, S_1)$, $G_2 = (\Sigma, V_2, P_2, S_2)$ with $V_1 \cap V_2 = \emptyset$.

Let $G = (\Sigma, V, P, S)$ with $V = V_1 \cup V_2 \cup \{S\}$, and

$P = P_1 \cup P_2 \cup \{ S \rightarrow S_1 \mid S_2 \}$. Then $L(G) = L(G_1) \cup L(G_2)$.

Concatenation: If L_1 and L_2 are CFLs, so is $L_1 L_2$.

Let $G_1 = (\Sigma, V_1, P_1, S_1)$, $G_2 = (\Sigma, V_2, P_2, S_2)$ with $V_1 \cap V_2 = \emptyset$.

Let $G = (\Sigma, V, P, S)$ with $V = V_1 \cup V_2 \cup \{S\}$, and

$P = P_1 \cup P_2 \cup \{ S \rightarrow S_1 S_2 \}$. Then $L(G) = L(G_1) L(G_2)$.

Kleene Star: If L_1 is a CFL, so is L_1*.

Let $G_1 = (\Sigma, V_1, P_1, S_1)$.

Let $G = (\Sigma, V, P, S)$ with $V = V_1 \cup \{S\}$, and

$P = P_1 \cup \{ S \rightarrow \varepsilon \mid S S_1 \}$. Then $L(G) = L(G_1)^*$.

Closure Properties for CFL

CFLs are **not** closed under intersection or complement

Intersection: $L_1 = \{ 0^i1^j0^k \mid i=j \}$ & $L_1 = \{ 0^i1^j0^k \mid j=k \}$ are CFLs. But it turns out that $L_1 \cap L_2 = \{ 0^i1^j0^k \mid i=j=k \}$ is not a CFL!

Complement: If CFLs were to be closed under complementation, since they are already closed under union, they would have been closed under intersection!

Grammars

Rewriting rules for generating strings from a "seed"

In an "unrestricted" grammar, the rules are of the form $\alpha \rightarrow \beta$ where $\alpha, \beta \in (\Sigma \cup V)^*$

Context-Free Grammar: Rewriting rules apply to individual variables (with no "context")